Algebraic stacks #10

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Moduli of elliptic curves & modular forms (lecture by Jan)

§ Moduli stack of elliptic curves

Fibered category:

$$\mathcal{M}_{1,1} \qquad \text{obj} = \left\{ \begin{array}{c} E \\ e \uparrow\downarrow \\ S \end{array} \text{ smooth proper}: \ e \text{ section}, \ \forall \operatorname{Spec} k \xrightarrow{s} S: \underbrace{(E_s, e(s))}_{\text{elliptic curve}} \right\}$$

$$\downarrow$$

$$Sch \qquad \text{mor} = \left\{ \text{cartesian squares} \quad \begin{tikzcd} E' \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\square"] & E \arrow[d, shift left] \\ S' \arrow[r] \arrow[u, shift left, "e'"] & S \arrow[u, shift left, "e"'] \end{tikzcd} \quad \text{commuting w/ } e, e' \right\}$$

$$\overline{\mathcal{M}}_{1,1} \qquad \text{obj} = \left\{ \begin{array}{c} E \\ e \uparrow\downarrow \\ S \end{array} \text{ flat + proper} \quad : \quad \text{stable ell curve} \right)$$

Fact: $\mathcal{M}_{1,1}$ DM $/\mathbb{Z}$. Will prove this when 6 invertible.

A smooth presentation when 6 invertible $(6 \in k^\times)$

Weierstrass family:

$$\begin{array}{c} \mathcal{E} \subset \mathbb{P}^2_k \times (\mathbb{A}^2 \setminus \Delta) \\ e \uparrow\downarrow \\ \mathbb{A}^2 \setminus V(\Delta) = \operatorname{Spec} k[u, v, \Delta^{-1}] \end{array}$$

$\mathcal{E}$ Zariski-closure of $y^2 = 4x^3 - ux - v$

$\Delta = u^3 - 27v^2$

$$e = [0:1:0]$$

This gives a smooth surjective map $\mathbb{A}^2 \setminus V(\Delta) \longrightarrow \mathcal{M}_{1,1}$.

There is an action of $\mathbb{G}_m$ on $\mathbb{A}^2_u$ given by $\lambda.(u,v) = (\lambda^4 u, \lambda^6 v)$

$$\mathcal{M}_{1,1} = \left[\mathbb{A}^2 \setminus V(\Delta) / \mathbb{G}_m\right]$$

Stabilizers: $u \neq 0, v \neq 0 \Rightarrow \mu_2$ $\Rightarrow$ étale stabilizers

$u \neq 0, v = 0 \Rightarrow \mu_4$ in char $\neq 2,3$

$u = 0, v \neq 0 \Rightarrow \mu_6$ $\Rightarrow$ DM.

__Rmk__: There is $\omega \in \Omega^1(\mathcal{E})$ s.th $\omega = \frac{dx}{y}$ on Zariski open $y \neq 0$.

$$\omega \in \Gamma(\mathbb{A}^2_u \setminus \Delta,\ p_* \Omega_{\mathcal{E}/\mathbb{A}^2 \setminus \Delta})$$

§ Quasi-coherent sheaves on DM-stacks

Let $\mathcal{M}$ be DM stack

__Def__: A __quasi-coherent sheaf__ $\mathcal{F}$ on $\mathcal{M}$ is:

- $\varphi^* \mathcal{F} \in \mathrm{QCoh}(S_{zar})$ for all $S \xrightarrow{\varphi} \mathcal{M}$, S scheme
- For every 2-commutative diagram
$$\begin{tikzcd} S' \arrow[rr, "f"] \arrow[dr, "\varphi'"'] & \arrow[d, phantom, "\Swarrow"] & S \arrow[dl, "\varphi"] \\ & \mathcal{M} & \end{tikzcd}$$
an isomorphism
$$\alpha_f : f^* \varphi^* \mathcal{F} \longrightarrow \varphi'^* \mathcal{F}$$
satisfying the cocycle condition.

A <u>global section</u> s of $\mathcal{F}$ is

- $\varphi^* s \in \Gamma(S, \varphi^* \mathcal{F})$ for each $\varphi : S \longrightarrow M$.

s.th. for every 2-commutative diagram as above $\alpha_f(f^* \varphi^* s) = \varphi'^* s$

§ The Hodge bundle

Given $\varphi : S \longrightarrow \mathcal{M}_{1,1}$ corresponding to family of ell. curves $\begin{array}{c} E \\ e \uparrow \downarrow p \\ S \end{array}$.

By cohomology and base change:

- $\varphi^* \mathcal{F} := p_* \Omega^1_E$ rank 1 vector bundle
- for every $S' \xrightarrow{f} S$, the natural map $\alpha_f : f^* p_* \Omega^1_E \longrightarrow p'_* \Omega^1_{E'/S'}$ is an isomorphism

$\rightsquigarrow$ quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}_{1,1}$ (the Hodge bundle)

(Also: $\mathcal{F} = p_* \Omega^1_{\mathcal{E}/\mathcal{M}_{1,1}}$ where $\mathcal{E} \longrightarrow \mathcal{M}_{1,1}$ univ. family.)

<u>Rmk</u>: $\omega \in \Gamma(\mathbb{A}^2 \setminus \Delta, p_* \Omega^1_{\mathcal{E}/\mathbb{A}^2 \setminus \Delta})$ (prev. rmk) shows that Hodge bundle restricted to $\mathbb{A}^2 \setminus \Delta$ is trivial.

§ Modular forms as sections of Hodge bundle

Def: Let $k \in \mathbb{Z}$. A mod form of wt k is a holomorphic map $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying

(1) $f(\gamma\tau) = (c\tau + d)^k f(\tau) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$.

(2) f is bounded as $\operatorname{Im}(\tau) \to \infty$.

Analytification of $\mathcal{M}_{1,1}|_{\mathbb{C}}$ is a stack $\mathcal{M}_{1,1}^{an}$ over $\mathbb{C}$-analytic spaces.

Thm: $\mathcal{M}_{1,1}^{an} \simeq [\mathbb{H} / SL_2\mathbb{Z}]$. Étale presentation $\mathbb{H} \xrightarrow{\pi} \mathcal{M}_{1,1}^{an}$.

Analogously, there is a Hodge bundle $\mathcal{F}$ on $\mathcal{M}_{1,1}^{an}$.

Rmk: $\exists\ \mathbb{H} \xrightarrow{f} \mathbb{A}^2 \setminus \Delta$ given by Weierstrass $\wp$-function.* inducing iso on quotients by $SL_2\mathbb{Z}$ and $\mathbb{G}_m$ respectively, and:

$$f^* : \Gamma(\mathbb{A}^2 \setminus \Delta,\ p^*\mathcal{F}) \longrightarrow \Gamma(\mathbb{H},\ p^*\mathcal{F})$$
$$\omega \longmapsto \underbrace{(\tau \mapsto 2\pi i\, dz)}_{=\ \ddot{\omega}}$$

* Weierstrass $\wp$-function: $\wp(z,\tau) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}(\tau)\, z^{2n}$

Eisenstein series $G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \frac{1}{(m+n\tau)^{2k}}$

modular form of weight $2k$ for $k \geq 2$.

$f(\tau) = (g_2(\tau), g_3(\tau)) \qquad g_2 = 60G_4,\ g_3 = 140G_6$

$f(\gamma\tau) = \lambda . f(\tau)$ where $\lambda = c\tau + d$.

We have:

$$\begin{tikzcd}
 & \omega^{\otimes k} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\Gamma(\mathbb{H}, \mathbb{C}) \arrow[r, "\cong"] \arrow[d, phantom, "\cup"] & \Gamma(\mathbb{H}, \pi^*\mathcal{F}^{\otimes k}) \arrow[d, phantom, "\cup"] \\
\begin{pmatrix} \text{holo } f: \mathbb{H} \to \mathbb{C} \\ \text{satisfying } (1) \end{pmatrix} \arrow[r, "\cong"] \arrow[dd, phantom, "\bigcup"] & \Gamma(\mathbb{H}, \pi^*\mathcal{F}^{\otimes k})^{SL_2} \\
 & \Gamma(\mathcal{M}_{1,1}^{an}, \mathcal{F}^{\otimes k}) \arrow[u, "\cong"'] \\
\begin{pmatrix} \text{mod forms} \\ \text{of wt } k \end{pmatrix} \arrow[r, "\cong"] & \Gamma(\overline{\mathcal{M}}_{1,1}^{an}, \overline{\mathcal{F}}^{\otimes k}) \arrow[u, hook]
\end{tikzcd}$$

$$f \longmapsto f\omega^{\otimes k}$$

The $SL_2\mathbb{Z}$-action on $\pi^*\mathcal{F}^{\otimes k}$ is given by $(\gamma.\sigma)(\tau) = \dfrac{\sigma(\tau)}{(c\tau+d)^k}$

§ Algebraicity of modular forms

Thm: There is a $\mathbb{C}$-linear iso

$$\begin{array}{ccccc}
\Gamma(\overline{\mathcal{M}}_{1,1}|_{\mathbb{C}}, \overline{\mathcal{F}}^{\otimes k}) & \xrightarrow{\ \cong\ } & \Gamma(\overline{\mathcal{M}}_{1,1}^{an}, \overline{\mathcal{F}}^{\otimes k}) & = & \begin{pmatrix} \text{mod forms} \\ \text{of wt } k \end{pmatrix} \\
\cap & & \cap & & \\
\Gamma(\mathcal{M}_{1,1}|_{\mathbb{C}}, \mathcal{F}^{\otimes k}) & \subset & \Gamma(\mathcal{M}_{1,1}^{an}, \mathcal{F}^{\otimes k}) & = & \left\{ \begin{array}{l} \text{holo } f: \mathbb{H} \to \mathbb{C} \\ \text{satisfying } (1) \end{array} \right\} \\
\parallel & & & & \\
\begin{pmatrix} \text{weak mod forms} \\ \text{of wt } k \end{pmatrix} & = & \multicolumn{3}{l}{\left\{ \begin{array}{l} \text{holo } f: \mathbb{H} \to \mathbb{C} \\ \text{satisfying } (1) + \text{mero. at } i\infty \end{array} \right\}}
\end{array}$$

Behrend's trace formula (lecture by Sjoerd)

§ **Frobenius**: $X_0/\mathbb{F}_q$ scheme

Absolute Frobenius: $\mathrm{Frob}: X_0 \longrightarrow X_0$ identity on $|X_0|$, $(-)^q$ on $\mathcal{O}_{X_0}$

For $X := X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ define $F_q := \mathrm{Frob}_q \otimes \mathrm{id} : X \longrightarrow X$.
(relative or geometric Frobenius). No longer identity on $|X|$.

$$\begin{tikzcd} X \arrow[rr, "F_q"] \arrow[dr] & \circ & X \arrow[dl] \\ & \operatorname{Spec} \overline{\mathbb{F}}_q & \end{tikzcd}$$

Ex: $X_0 = \mathbb{A}^1_{\mathbb{F}_q}$. $X = \mathbb{A}^1_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q[t]$

$$\begin{aligned} F_q : \overline{\mathbb{F}}_q[t] &\longrightarrow \overline{\mathbb{F}}_q[t] \\ \Sigma a_i T^i &\longmapsto \Sigma a_i T^{qi} \\ (T - x^q) &\longleftarrow\!\shortmid (T - x) \end{aligned}$$

Let $\mathcal{X}_0/\mathbb{F}_q$ be a stack. For $S \in \mathrm{Sch}_{\mathbb{F}_q}$, let

$$\begin{aligned} \mathrm{Frob}_q(S) : \mathcal{X}(S) &\longrightarrow \mathcal{X}(S) \\ x &\longmapsto x \circ \mathrm{Frob}_q \end{aligned}$$

Since
$$\begin{tikzcd} S \arrow[r, "\mathrm{Frob}_q"] & S \arrow[r, "x"] & \mathcal{X}_0 \\ T \arrow[u] \arrow[r, "\mathrm{Frob}_q"] \arrow[ur, phantom, "\circ"] & T \arrow[u] & \end{tikzcd}$$
$\Rightarrow$ $\mathrm{Frob}_q : \mathcal{X}_0 \longrightarrow \mathcal{X}_0$ absolute Frobenius.

For $\mathcal{X} = \mathcal{X}_0 \otimes \overline{\mathbb{F}}_q$, $F_q := \mathrm{Frob}_q \otimes \mathrm{id}$.

§ ℓ-adic sheaves

Roughly, ℓ-adic sheaves on $\mathcal{X}$ is a (lisse-)étale sheaf of $\mathbb{Q}_\ell$-modules ($\ell \neq p$).

$\rightsquigarrow$ derived category $D_\ell(\mathcal{X})$.

Has a 6-functor formalism:

- For $f: \mathcal{X} \to \mathcal{Y}$, have $f_!, f_*: D_\ell(\mathcal{X}) \longrightarrow D_\ell(\mathcal{Y})$

  $f^!, f^*: D_\ell(\mathcal{Y}) \to D_\ell(\mathcal{X})$

- $- \otimes^L - : D_\ell(\mathcal{X}) \times D_\ell(\mathcal{X}) \longrightarrow D_\ell(\mathcal{X})$

  $R\mathcal{H}om(-,-)$: ——— " ———

If f structure map: $R^i f_* \mathcal{F} = H^i(\mathcal{X}, \mathcal{F})$

$\mathcal{X} \xrightarrow{f} \operatorname{Spec} \mathbb{F}_q$ $\quad R^i f_! \mathcal{F} = H^i_c(\mathcal{X}, \mathcal{F})$ compactly supported coh.

<u>Poincaré duality</u> ($\mathcal{X}$ smooth, dim d)

$$H^i_c(\mathcal{X}, \mathcal{F}) \cong H^{2d-i}(\mathcal{X}, \mathcal{F}^\vee)^\vee \otimes \mathbb{Q}_\ell(-d)$$

Frobenius action

(e.g. a constructible sheaf)

Suppose $\mathcal{F}$ is an ℓ-adic sheaf w/ $F_q^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. Then

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "F_q"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & \mathrm{Spec}(\overline{\mathbb{F}}_q) & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} H^i(\mathcal{X}, \mathcal{F}) \arrow[r, "\sim"] \arrow[dr] & H^i(\mathcal{X}, F_q^*\mathcal{F}) \arrow[d, "\wr"] \\ & H^i(\mathcal{X}, \mathcal{F}) \end{tikzcd}$$

This makes $H^i(\mathcal{X}, \mathcal{F})$ into a Galois rep. $\left(\mathrm{Gal}(\overline{\mathbb{F}}_q) = \langle \mathrm{Frob} \rangle\right)$

Weil conjectures: $\mathrm{Frob} \curvearrowright H^i(X, \mathbb{Q}_\ell)$ for X smooth & proper has eigenvalues of abs. value $q^{i/2}$.

Ex:
- $\mathbb{Q}_\ell$ constant sheaf (trivial Galois action)
- $\mathbb{Q}_\ell(k) := \mathbb{Q}_\ell$ as sheaves but $\mathrm{Frob}.v = q^{-k} v$.

§ Groupoid cardinality

Def: Let $\mathcal{C}$ be a groupoid. Then the groupoid cardinality is:

$$\#\mathcal{C} = \sum_{x \in [\mathcal{C}]} \frac{1}{\#\mathrm{Aut}(x)} \qquad \left([\mathcal{C}] = \pi_0 \mathcal{C}\right)$$

§ Trace formula

Thm (Trace formula, Behrend '03, Sun '10)
Let $\mathcal{X}_0$ be an alg. stack of finite type$/\mathbb{F}_q$. Then

$$\#\mathcal{X}_0(\mathbb{F}_q) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(F_q \mid H^i_c(\mathcal{X}, \mathbb{Q}_\ell))$$

More generally, for a constructible sheaf $\mathcal{F}$ (or ℓ-adic w/ Frob-action)

$$\sum_{x\in[\mathcal{X}_0(\mathbb{F}_q)]} \frac{\operatorname{Tr}(F_q \mid \mathcal{F}_x)}{\#\operatorname{Aut}(x)} = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(F_q \mid H^i_c(\mathcal{X}, \mathcal{F}))$$

Ex 1: $\mathcal{X} = BG$, G finite group

$$\begin{aligned}\mathcal{X}(\mathbb{F}_q) &= \{G\text{-torsors on } \mathbb{F}_q\} = G/\sim \quad (\text{conjugacy classes})\\ &= \{\operatorname{Spec}\mathbb{F}_q \longrightarrow BG\}\\ &= \{\hat{\mathbb{Z}} \longrightarrow G\}\end{aligned}$$

(orbit-stabilizer formula)

$$\text{LHS} = \#\mathcal{X}(\mathbb{F}_q) = \sum_{[g]\in G/\sim} \frac{1}{\#\operatorname{Stab}(g)} = \sum_{[g]\in G/\sim} \frac{\#\operatorname{orb}(g)}{|G|} = 1$$

- $\operatorname{Sh}_\ell(*) = \operatorname{Rep}_{\mathbb{Q}_\ell}(\operatorname{Gal}(\mathbb{F}_q))$
- $\operatorname{Sh}_\ell(BG) = \operatorname{Rep}^G_{\mathbb{Q}_\ell}(\operatorname{Gal}(\mathbb{F}_q))$

$$\begin{tikzcd} * \arrow[r, "p"] \arrow[dr, equal] & BG \arrow[d, "\pi"] \\ & * \end{tikzcd}$$

$p^*V = V$ (forget action)

$\pi^*V = V$ (w/ trivial action)

$\pi_* V = V^G$

$R\pi_*\mathbb{Q}_\ell = \pi_!\mathbb{Q}_\ell = \mathbb{Q}_\ell$

$$\text{RHS} = \operatorname{Tr}(F_q \mid \mathbb{Q}_\ell) = 1$$

Ex 2: $\mathcal{X} = B\mathbb{G}_m$, $\dim \mathcal{X} = -1$.

Lang's theorem $\Rightarrow B\mathbb{G}_m(\mathbb{F}_q) = \{*\}$

$$\text{LHS} = \# B\mathbb{G}_m(\mathbb{F}_q) = \frac{1}{\# \mathbb{G}_m(\mathbb{F}_q)} = \frac{1}{\# \mathbb{F}_q^*} = \frac{1}{q-1}$$

<u>ℓ-adic = sing over $\mathbb{C}$:</u>

$$\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \simeq S^1 \qquad BS^1 \simeq \mathbb{C}P^\infty$$

$$H^i(\mathbb{C}P^\infty; \mathbb{C}) = \underset{0}{\mathbb{C}} \oplus \underset{1}{0} \oplus \underset{2}{\mathbb{C}(-1)} \oplus \underset{3}{0} \oplus \underset{4}{\mathbb{C}(-2)} \oplus \cdots$$

$$H^i(\mathcal{X}, \mathbb{Q}_\ell) = \underset{0}{\mathbb{Q}_\ell} \oplus \underset{1}{0} \oplus \underset{2}{\mathbb{Q}_\ell(-1)} \oplus \underset{3}{0} \oplus \underset{4}{\mathbb{Q}_\ell(-2)} \oplus \cdots$$

Duality gives:

$$H^i_c(\mathcal{X}, \mathbb{Q}_\ell) = \cdots \oplus \underset{-4}{\overset{q^{-2}}{\mathbb{Q}_\ell(2)}} \oplus \underset{-3}{0} \oplus \underset{-2}{\overset{\text{Frob} = q^{-1}}{\mathbb{Q}_\ell(1)}} \oplus \underset{-1}{0} \oplus \underset{0}{0}$$

$$\text{RHS} = \sum_{i \geq 1} q^{-i} = \frac{1}{q} \cdot \frac{1}{1 - q^{-1}} = \frac{1}{q-1}$$

Ex 3: $\mathcal{X} = \mathcal{M}_{1,1}|_{\mathbb{F}_q}$, $\quad \mathcal{E} \xrightarrow{\pi} \mathcal{M}_{1,1}$ universal curve

$$\mathbb{V}_k := \mathrm{Sym}^k(R^1\pi_* \mathbb{Q}_\ell)$$

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[d] \\ \operatorname{Spec} \mathbb{F}_q \arrow[r, "x"] & \mathcal{M}_{1,1} \end{tikzcd}$$

$$x^* \mathbb{V}_1 = H^1_{et}(E, \mathbb{Q}_\ell) =: V_\ell(E)$$

Frob eigenvalues $\pi_E, \overline{\pi_E}$

$$\text{LHS} = \sum_{[E]/\mathbb{F}_q} \frac{h_k(\pi_E, \overline{\pi_E})}{\# \mathrm{Aut}(E)}$$

$\boxed{\text{RHS}}$ $\quad H^i(\mathcal{M}_{1,1}, \mathbb{V}_k) = \begin{cases} 0 & i \neq 1 \\ \underbrace{S[k+2]}_{\substack{\dim = 2\dim S_{k+2} \\ \text{wt } k+1}} \oplus \underbrace{E[k+2]}_{\substack{\dim 1 \\ \text{wt } 2k+2}} & i = 1 \end{cases}$ $\quad \cong \mathbb{Q}_\ell(-k-1)$

Duality gives:

$$H^i_c(\mathcal{M}_{1,1}, \mathbb{V}_k) = S[k+2] \oplus \mathbb{Q}_\ell$$

$$\Rightarrow \text{RHS} = \mathrm{Tr}(T_p | S_{k+2}) - 1$$