Algebraic stacks #4

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§ More on groupoids / CFGs: (2,1)-categories

§ Descent

§ Stacks (examples)

§ Category of groupoids

- **Cat** is a **2-category**:
    - objects = categories $\qquad (\mathcal{C}, \mathcal{D})$
    - morphism = functors $\qquad (F, G)$
    - 2-morphisms = natural transformations $\qquad (\alpha)$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] & \mathcal{D} \arrow[Rightarrow, from=G, to=F, "\alpha"] \end{tikzcd}$$

$\mathrm{Map}_{\mathrm{Grpd}}(\mathcal{C}, \mathcal{D})$ is a category: obj = functors $\quad (F, G)$
arrows = nat. tfm. $\quad (\alpha)$

- **Grpd** is a **(2,1)-category**:
    - objects = categories w/ all arrows invertible
    - morphism = functors
    - 2-morphisms = natural transformations = natural isomorphisms

$\qquad$ b/c arrows in $\mathcal{D}$ are invertible

so all 2-morphisms invertible

Differently put: $\mathrm{Map}_{\mathrm{Grpd}}(\mathcal{C}, \mathcal{D})$ is a groupoid

Ex: $BG = \begin{tikzcd} \bullet \arrow[loop above, "g_1"] \arrow[loop right, "g_2"] \arrow[loop below] \end{tikzcd} \qquad \mathrm{Hom}(\bullet, \bullet) = G.$

$H < G$ subgroup. $\qquad \mathrm{Hom}(BG, BH) = $ groupoid

Exc: Describe this groupoid

§ Category of CFG

$$\begin{tikzcd} \mathcal{C}_1 \arrow[dr, "\pi_1"'] & & \mathcal{C}_2 \arrow[dl, "\pi_2"] \\ & \mathcal{D} & \end{tikzcd}$$

fibered categories $\Rightarrow$ $\mathrm{MOR}_{\mathcal{D}}(\mathcal{C}_1, \mathcal{C}_2)$ category

$\Rightarrow$ Categories fibered over $\mathcal{D}$ is a 2-category.

Prop [0, 3.4.3] If $\mathcal{C}_2$ CFG, then $\mathrm{MOR}_{\mathcal{D}}(\mathcal{C}_1, \mathcal{C}_2)$ groupoid

pf: Given $\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{C}_2 \end{tikzcd}$, then by definition α is base-preserving

so $\alpha(x) : f(x) \to g(x)$ lies over $id_{\pi_1(x)} \in \mathcal{D}$

i.e., $\alpha(x)$ arrow in $\mathcal{C}_2(\pi_1(x))$, hence invertible $\forall x \in \mathcal{C}_1$. $\square$

$\Rightarrow$ $CFG/_{\mathcal{D}}$ is a $(2,1)$-category.

Prop [0, 3.4.12] The 2-fiber product of CFG's exists and is a CFG.

§ Descent theory

Motivation: X scheme, $X = \bigcup U_i$ open covering or $\coprod U_i \to X$ étale covering

Ex 0: $\mathcal{F}$ sheaf of sets on X means:

$$\mathcal{F}(X) \xrightarrow{\varphi} \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_X U_j)$$

exact seq of sets (= equalizer diagram). Entails 2 things:

- injectivity of φ: $s|_{U_i} = t|_{U_i} \Rightarrow s = t \quad \forall s,t \in \mathcal{F}(X)$
- exactness in middle: (s_i) s.th. $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ glues to $s \in \mathcal{F}(X)$.

Ex -1: If $\mathcal{F}$ takes values in $\{\emptyset, *\}$ ("$\mathcal{F}$ sheaf of (-1)-categories") either no section or 1 section.
$\mathcal{F}$ sheaf equivalent to

$$\mathcal{F}(X) = \prod \mathcal{F}(U_i)$$

(If $\exists$ section locally, then $\exists$ global section.)

Rmk: Injectivity in Ex 0 $\Leftrightarrow$ Presheaf $\mathrm{Isom}(s,t): U \mapsto \mathrm{Isom}(s|_U, t|_U)$ is a sheaf

$$\mathrm{Isom}(s|_U, t|_U) = \begin{cases} * & \text{if } s|_U = t|_U \\ \emptyset & \text{o/w} \end{cases}$$

When $\mathcal{F}$ presheaf of categories = pseudo-functor ($\Leftrightarrow$ fibered category) gluing is more complicated.

Ex 1: Consider the following presheaves of categories

$F: Op(X)^{op} \longrightarrow \mathcal{C}at$ (or $Et(X)^{op} \longrightarrow \mathcal{C}at$, or $Sch^{op} \to \mathcal{C}at$)

$Shv: \quad U \longmapsto Shv(U) = $ sheaves on U.

$Mod_{\mathcal{O}_X}: \quad U \longmapsto Mod(\mathcal{O}_U) = $ sheaves of $\mathcal{O}_U$-modules

$QCoh: \quad U \longmapsto QCoh(U) = $ qcoh sheaves on U

$Sch: \quad U \longmapsto Sch_{/U} = $ schemes $Y \to U$

$Aff: \quad U \longmapsto Aff_{/U} = $ affine schemes $Y \to U$

$QAff:$ $\qquad\qquad\qquad\qquad\qquad = $ qaffine

To glue objects $(E_i \in F(U_i))_{i\in I}$ along covering $\coprod U_i \longrightarrow X$ we need:

gluing datum
$\quad$ ‖

- descent datum, i.e., isomorphisms $\pi_1^* E_i \xrightarrow[\alpha_{ij}]{\cong} \pi_2^* E_j$ in $F(U_i \times U_j)$

$U_i \times U_j \xrightarrow{\pi_1} U_i$ etc

such that cocycle condition holds:

$$\alpha_{jh} \circ \alpha_{ij} = \alpha_{ih} \quad \text{in } F(U_i \times U_j \times U_h)$$

To be precise: [O, 4.2.3]

$$\begin{tikzcd}
 & \pi_{12}^*\pi_1^*E_j \arrow[r, phantom, "\overset{can}{\cong}"] & \pi_{23}^*\pi_1^*E_j \arrow[dr, "\pi_{23}^*\alpha_{jh}"] & \\
\pi_{12}^*\pi_1^*E_i \arrow[ur, "\pi_1^*\alpha_{ij}", "\cong"'] \arrow[d, phantom, "can\ \cong"] & & & \pi_{23}^*\pi_2^*E_h \arrow[d, phantom, "\cong\ can"] \\
\pi_{31}^*\pi_2^*E_i & & & \pi_{31}^*\pi_1^*E_h \arrow[lll, "\pi_{31}^*\pi_1^*\alpha_{hi}"]
\end{tikzcd}$$

in $F(U_i \times_X U_j \times_X U_h)$ $\qquad$ $U_i \times_X U_j \times_X U_h \xrightarrow{\pi_{12}} U_i \times_X U_j$ etc

$$\text{"} F(X) = \lim \left(\prod F(U_i) \rightrightarrows \prod F(U_i \times_X U_j) \; \substack{\rightarrow\\\rightarrow\\\rightarrow} \; \prod F(U_i \times_X U_j \times_X U_k) \right) \text{"}$$

limit of categories.

Down-to-earth description:

<u>Def</u>: Category of descent data:

$$F(\{U_i \longrightarrow X\}_i) = \text{category}:$$

$$\text{obj} = \left\{ (E_i)_i, (\alpha_{ij})_{ij} \; : \; E_i \in F(U_i), \; \alpha_{ij} : \pi_1^* E_i \xrightarrow{\simeq} \pi_2^* E_j \right.$$
$$\left. \text{s.th. satisfies cocycle condition} \right\}$$

$$\text{arrows}: \; \mathrm{Map}\left((E_i, \alpha_{ij}), (E_i', \alpha_{ij}') \right)$$
$$= \left\{ f_i : E_i \longrightarrow E_i' \; : \; \alpha_{ij}' \circ \pi_1^* f_i = \pi_2^* f_j \circ \alpha_{ij} \right\}$$

<u>Def</u>: There is a natural functor

$$F(X) \xrightarrow{\varepsilon} F(\{U_i \longrightarrow X\})$$

An object in the essential image is called an effective descent datum. $\{U_i \longrightarrow X\}$ is of effective descent for F if ε equivalence.
(descent) (fully faithful)

$(\Leftrightarrow \mathcal{C} \longrightarrow D$ fibered cat$)$

<u>Def</u>: [O, 4.6.1] $F : D^{op} \longrightarrow \mathrm{Cat}$ is a stack (or a sheaf) for the Groth. topology τ if every covering in τ of effective descent for F.

Rmk: When $F: \mathcal{D}^{op} \to \mathrm{Set}$ presheaf of <u>sets</u>, then
$$F(\{U_i \to X\}) = \{ s_i \in F(U_i) : s_i|_{U_{ij}} = s_j|_{U_{ij}} \}$$
so becomes usual sheaf condition.

§ Examples of stacks

Fact: All examples above $\mathrm{Sch}^{op} \to \mathrm{Cat}$ are stacks for the Zariski topology

Fact: All examples, <u>except</u> Sch, are stacks for the étale and even fppf (and fpqc) topologies

For Shv, $\mathrm{Mod}_{\mathcal{O}_X}$, QCoh, see [0, 4.2.10, 4.2.13, 4.3].
This leads to proofs for Aff, QAff, see [0, 4.4.7, 4.4.15].

Rmk: "Correction" for Sch: consider instead one of:

$$\mathrm{Pol}: U \longmapsto \{ X \xrightarrow{f} U, \mathcal{L} : X \text{ scheme}, \mathcal{L} \text{ f-ample l.b} \}$$
$$\mathrm{AlgSp}: U \longmapsto \{ X \to U : X \text{ algebraic space} \}$$

These are stacks for the étale and even fppf topologies.

(for Pol see [0, 4.4.10])

§ Example: closed subschemes

<u>Ex</u> [0, 4.4.1] (closed subschemes)

The functor $Cl : Sch^{op} \longrightarrow Set$

$$X \longmapsto \{ Z \hookrightarrow X \text{ closed subsch} \}$$

is a sheaf.

<u>pf</u>: $Cl(X) = \{ \mathcal{I} \subset \mathcal{O}_X : \text{qcoh ideal} \}$. (set)

set-valued functor   Cat-valued functor

$Cl(X) \longrightarrow QCoh(X)$ fully faithful

$\mathcal{I} \subset \mathcal{O}_X \longmapsto \mathcal{I}$

$$\{ (\mathcal{I}_i \subset \mathcal{O}_{U_i}) : \mathcal{I}_i|_{U_{ij}} = \mathcal{I}_j|_{U_{ij}} \}$$

$$\|$$

$$\Rightarrow \quad \begin{array}{ccc} Cl(X) & \xrightarrow{\varepsilon} & Cl(\{U_i \to X\}) \\ \cap & & \cap \\ QCoh(X) & \xrightarrow[\varepsilon_{QC}]{\cong} & QCoh(\{U_i \to X\}) \end{array} \quad \text{fully faithful}$$

An object in $Cl(\{U_i \to X\})$ gives, since ε_{QC} ess. surj., a qcoh sheaf $\mathcal{E}$ on X. The inclusions $\mathcal{I}_i \to \mathcal{O}_{U_i}$ gives $\mathcal{E} \to \mathcal{O}_X$ (since ε_{QC} fully faithful).
Since $\coprod U_i \to X$ <u>faithfully flat</u>, injectivity of $\mathcal{I}_i \to \mathcal{O}_{U_i}$ gives injectivity of $\mathcal{E} \to \mathcal{O}_X$. $\square$

§ Example: vector bundles

Ex: (Vector bundles) The functor

$$VB : Sch^{op} \longrightarrow Cat$$
$$X \longmapsto VB(X) = \{\mathcal{E} \text{ loc free sheaf of finite rank}\}$$

is a stack.

pf: Again $VB(X) \longrightarrow QC(X)$ f.f. $\Rightarrow$ $VB(X) \to VB(\{U_i \to X\})$ f.f.

Ess. surj follows from

$$\mathcal{E} \in QCoh(X), \quad \mathcal{E}|_{U_i} \text{ v.b. } \forall i \quad \Longrightarrow \quad \mathcal{E} \text{ v.b.}$$

which follows from faithful flatness. (see lemma below) $\square$

Lemma: $A \longrightarrow B$ faithfully flat. $M \in Mod_A$.

$M \otimes_A B$ loc free of finite rank $\Rightarrow$ M loc free.

§ Example: moduli of curves

Ex. $\mathcal{M}_g : Sch^{op} \to Cat$ is a stack for $g \geq 2$.

pf. Have functor $\mathcal{M}_g \longrightarrow \mathcal{P}ol$.

$$\begin{array}{c} C \\ f\downarrow \\ T \end{array} \longmapsto \left(\begin{array}{c} C \\ \downarrow f \\ T \end{array}, \Omega_f \right)$$ (canonically) polarized scheme

Not fully faithful (more automorphisms due to line bundle)

Given covering $\{U_i \to T\}$, families of curves $\begin{array}{c} C_i \\ \downarrow \\ U_i \end{array}$

gluing data
$$\begin{tikzcd} C_i|_{U_{ij}} \arrow[rr, "\alpha_{ij}", "\cong"'] \arrow[dr] & & C_j|_{U_{ij}} \arrow[dl] \\ & U_{ij} & \end{tikzcd}$$
(satisfying cocycle cond) $\Rightarrow$ gluing datum for (C_i, Ω_{C_i})

so glues to polarized scheme $\left(\begin{array}{c} C \\ \downarrow \\ T \end{array}, \mathcal{L} \right)$. Forgetting pol'n gives $(C \to T) \in \mathcal{M}_g$

Lemma: If $C \times_T U_i \to U_i$ smooth/proper/.. then so is $C \to T$.
(f. flat descent)

Rmk: For $g=0$, can use $\Omega_f^\vee$ instead.
For $g=1$, $\Omega_f = 0$ and $\mathcal{M}_1$ not a stack as defined. Need

$$\mathcal{M}_1(T) = \left\{ \begin{array}{c} C \\ \downarrow f \\ T \end{array} : C \text{ algebraic space}, f \text{ smooth, proper} \right\}$$
fibers genus 1 curves

Rmk: For $\mathcal{M}_{g,n}$, $3g-3+n \geq 0$, instead use $\Omega_{C/T}(\sum_{i=1}^n D_i)$

where $\begin{array}{c} C \\ s_i \uparrow\downarrow \\ T \end{array}$ family of curve w/ n disjoint sections $s_1, \ldots, s_n$ and
$D_i = s_i(T)$ divisor.