Homework exercises for lecture #2

To be handed in on paper on Feb 11 (lecture #3) or earlier by email.

Throughout X denotes a complex manifold and $x \in X$ denotes a point.

(1) Suppose that X is connected. Let f be a non-zero holomorphic function. Show that $Z(f) = \{x \in X \,:\, f(x) = 0\}$ is nowhere dense[1].

(2) Show that the ring $\mathcal{O}_{X,x}$ is an integral domain.

(3) Let $K(X)$ denote the ring of meromorphic functions on X.
   (a) Show that if X is connected, then $K(X)$ is a field.
   (b) What happens if X is not connected?
   (c) What is the relation between the fraction field of the integral domain $\mathcal{O}_{X,x}$ and $K(X)$?

If you need a hint, see next page.

[1] A subset $A \subseteq X$ is *dense* if $\overline{A} = X$. A subset $A \subseteq X$ is *nowhere dense* if $A \cap U \subseteq U$ is not dense for any (non-empty) open $U \subseteq X$. Equivalently, $\overline{A}$ has empty interior.

Hint for (2)*:* This can be proven both analytically and algebraically. An analytic approach could be to use (1). An algebraic approach could use a suitable order on monomials.

Hint for (3a)*:* It could be useful to use (1).