Homework problems for lecture #3

To be handed in on paper on Feb 18 (lecture #4) or earlier by email.

Functions on a complex torus. Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ be a lattice in $\mathbb{C}$, i.e., $\omega_1, \omega_2 \in \mathbb{C}$ are such that Λ spans $\mathbb{C} = \mathbb{R}^2$ as a real vector space. Recall that $X = \mathbb{C}/\Lambda$ is a complex torus. A holomorphic/meromorphic function on X is thus the same as a *doubly periodic* holomorphic/meromorphic function on $\mathbb{C}$, that is, a function such that $f(z+\omega) = f(z)$ for all $\omega \in \Lambda$.

Weierstrass function. The *Weierstrass function* is the function $\mathbb{C} \smallsetminus \Lambda \to \mathbb{C}$ given by:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

You do not have to prove the following facts.

(i) The series above converges absolutely and uniformly on compact subsets. More precisely, for any $R > 0$, there is a finite subset $\Lambda_R \subset \Lambda$ such that

$$\wp_R(z) = \sum_{\omega \in \Lambda \smallsetminus \Lambda_R} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is uniformly absolutely-convergent on $B_R(0)$.

(ii) It follows that $\wp_R(z)$ is holomorphic on $B_R(0)$.

Homework problems.

(1) (a) Show that $\wp(z)$ is an even meromorphic function and that $\wp'(z)$ is an odd meromorphic function.
(b) Show that $\wp'(z)$ is doubly periodic.
(c) Show that $\wp(z)$ is doubly periodic.
(d) Show that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

for some unique constants $g_2, g_3 \in \mathbb{C}$ (depending on Λ).

(2) We let $\mathbb{P}^2 = \{(x_0 : x_1 : x_2)\}$. Consider the map $f\colon \mathbb{C} \to \mathbb{P}^2$ defined by:

$$f(z) = \begin{cases} (\wp(z) : \wp'(z) : 1) & \text{if } z \notin \Lambda \\ (0:1:0) & \text{if } z \in \Lambda \end{cases}$$

By (1b) and (1c), this induces a map $g\colon \mathbb{C}/\Lambda \to \mathbb{P}^2$.
(a) Show that g is holomorphic.
(b) Show that the image of g is contained in the analytic subspace $E = Z(p) \subset \mathbb{P}^2$ where p is the homogeneous polynomial

$$p(x_0, x_1, x_2) = x_1^2 x_2 - (4x_0^3 - g_2 x_0 x_2^2 - g_3 x_2^3)$$

It can also be shown that g is injective with image E but you don't need to prove that.

If you need a hint, see next page.

Hint for (1c)*:* Prove this using (1a) and (1b). It is not immediate from the series expression of $\wp(z)$.

Hint for (1d)*:* Choose the constant g_2 such that $g_3(z) := 4\wp(z)^3 - \wp'(z)^2 - g_2\wp(z)$ becomes holomorphic in a neighborhood of 0 and then show that $g_3(z)$ is constant.

Hint for (2a)*:* It is enough to show that f is holomorphic over the two standard opens $U_2 = \{x_2 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$.