28/01/20

- Holomorphic functions

<u>Def</u>: Let $U \subseteq \mathbb{C}$ open. A function $U \longrightarrow \mathbb{C}$ is holomorphic at $x \in U$ if $\forall z$ in a nbhd of x $\quad f'(z) = \lim_{|h| \to 0} \frac{f(z+h) - f(z)}{h}$ exists.

<u>Rem</u> (equivalent definitions):

- Cauchy-Riemann equations: if $f(x+iy) = u(x,y) + i v(x,y)$
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
with all partial derivatives continuous

- $$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$
These two operators have the properties:
$$\frac{\partial}{\partial z}(z) = 1, \quad \frac{\partial}{\partial z}(\bar z) = 0, \quad \frac{\partial}{\partial \bar z}(z) = 0, \quad \frac{\partial}{\partial \bar z}(\bar z) = 1$$
indeed, if $z = x+iy \Rightarrow \bar z = x - iy$
so that one can write $\quad x = \frac{1}{2}(z+\bar z)$ and $y = \frac{-i}{2}(z - \bar z)$

Notice that: $\quad$ CR-eq $\Leftrightarrow \frac{\partial f}{\partial \bar z} = 0$

- Cauchy's integral formula:
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z - z_0}\, dz$$
$\leftarrow$ holomorphic functions can be evaluated in one point by integrating around a circle

- Analytic functions: $\forall z_0 \in U \; \exists \varepsilon > 0$ s.t $f|_{B_\varepsilon(z_0)}$ is
$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m$$
which is convergent $\quad$ holomorphic $\Rightarrow$ analytic

<u>Obs</u>: holomorphic $\Rightarrow C^\infty$ (it is possible to notice looking at the power series, for example)

- Algebraic functions

<u>Def</u>: Let $U \subseteq \mathbb{C}^n$ (Zariski-)open. A function $f : U \longrightarrow \mathbb{C}^m$ is "algebraic" (regular or polynomial) if $f(z) = \frac{p(z)}{q(z)}$ where $q(z) \neq 0$ on U, and $p(z), q(z)$ polynomial.

<u>Note</u>: It could be called rational, but in literature ~~polynom~~ rational functions might not be defined everywhere.

- Subspace of $\mathbb{C}^n$

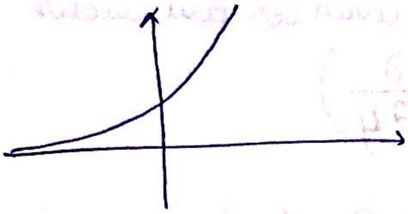
<u>Def</u>: An analytic subspace $X \subseteq \mathbb{C}^n$ is a closed subset "locally cut out by holomorphic functions", i.e $\forall x \in X \; \exists$ nbhd U of x and $f_1, \ldots f_r$ holomorphic functions on U s.t
$$X \cap U = Z(f_1, \ldots f_r) := \{ z \in U \mid f_1(z) = \_\_\_ = f_r(z) = 0 \} =$$
$$= V(f_1, \ldots f_r) \quad \text{"vanishing locally"}$$

__Def__: An algebraic subspace (subvariety) $X \subseteq \mathbb{C}^n$ is a closed subset __locally__ cut out by algebraic functions
↑ Zariski sense

In other words, $\exists\, p_1, \dots p_r$ polynomials s.t

$X = Z(p_1, \dots p_r)$

ex: $Z(y - e^x) \subseteq \mathbb{C}^2$ analytic but not algebraic subset



← there are no polynomials whose zeros describe this curve.

ex: $Z(y - x^2) \subseteq \mathbb{C}^2$ algebraic subset

__Note__: Of course every algebraic subset is analytic, but not vice versa

- Subspace of $\mathbb{C}^1$

$X = Z(p_1, \dots p_r) = \bigcap_{i=1}^{r} Z(p_i) = \{x_1, \dots x_m\}$ $\quad p_i(z) \in \mathbb{C}[z]$ $\quad p_i(z) = (z - z_1) \dots (z - z_m)$

$\Rightarrow Z(p_i) = \{z_1, \dots z_m\}$

$Z(1) = \emptyset$, $Z(0) = \mathbb{C}$ ← except for these, all the algebraic subspaces of $\mathbb{C}^1$ have a finite number of points

On the analytic side instead:
suppose U connected and f non-zero holomorphic fun on U, then $Z(f)$ is discrete \$

__Note__: discrete = every point has a open nbhd ~~that is open and~~ containing just that point
= every point is both open and closed

__pf__ $z_0 \in Z(f)$. Locally at z_0: $f(z) = \sum a_m (z - z_0)^m = (z - z_0)^d \underbrace{\sum b_m (z - z_0)}_{=: g(z)}$ / with $b_0 \neq 0$

$f \neq 0$ in the entire connected comp. where z_0 is m

$\Rightarrow g(z)$ non-vanishing in a nbhd $B_\varepsilon(z_0)$ of z_0

$\Rightarrow Z(f) = Z((z - z_0)^d) = \{z_0\}$ where do we use U connected?

possibly $d = 0$, $b_m = a_m$?

__Rem__: Any discrete subset X is analytic by def (locally at $z_0 \in X$, $X = Z(z - z_0)$)

ex: $Z(\sin z) = \{m\pi \mid m \in \mathbb{Z}\}$

__Note__: There are more analytic subspaces than algebraic subspaces.

We can conclude that $X = \mathbb{C}^1$ or X = discrete subset

$\Rightarrow$ compact analytic subspaces of $\mathbb{C}^1$ are finite subsets = compact algebraic subspaces.

This is also true for $\mathbb{C}^m$.

So $\mathbb{C}^m$ behaves well in this context but it is not very interesting, so we will shift to $\mathbb{P}^m_{\mathbb{C}}$.

• ~~Recall~~ Recall (Whitney embedding th): M real manifold $\Rightarrow \exists N >> 0$ s.t $M \hookrightarrow \mathbb{R}^N$.

This is "completely false" for compact complex manifolds.

Question: when can we embed complex manifold X in $\mathbb{P}^m_{\mathbb{C}}$?
(is going to require line bundles)

Th (Chow): P is an algebraic compact space (= proper scheme $/\mathbb{C}$), then every analytic subspace of P is algebraic.

Note: We need several ingredients in order to understand this th.

• Manifolds

Def: A topological ~~space~~ manifold is a top. space M (Hausdorff, 2nd countable) s.t $\exists$ open cover $M = \bigcup U_i$, $U_i \cong \mathbb{R}^m$ (homeomorphism)

Def: A C^∞-manifold (= differentiable m = M) with an atlas is a top. space M ~~that admits~~ with an C^∞-atlas, i.e $\exists$ open cover $M = \bigcup U_i$ together with $\varphi_i : U_i \longrightarrow \mathbb{R}^m$ homeomorphism s.t transition functions are C^∞:

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ \varphi_i(U_i \cap U_j) \arrow[rr, "\varphi_j \circ \varphi_i^{-1}"'] & & \varphi_j(U_i \cap U_j) \end{tikzcd}$$

called transition map.

make sense because they are maps between $\mathbb{R}^m$ (*)

Rem: Could replace $\mathbb{R}^m$ with $V_i \subseteq \mathbb{R}^m$ open

Def: Two atlases are equivalent (define the same manifold) if they are compatible, i.e, the union of the atlases is an atlas. (*)

Def: A m-dimensional complex manifold is a top. space M that admits a holomorphic atlas: $M = \bigcup U_i$, $\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{C}^m$ s.t

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j) \text{ holomorphic.}$$

Rem: An m-dim complex manifold is $2m$-dim C^∞-manifold

(*) Def: A differentiable manifold is a top. space with an equivalent class of atlases.

same correction for the complex manifold.

- Sheaves

Def: Let M be a manifold. For $U \subseteq M$ open, $C^\infty_M(U) = \{ f : U \to \mathbb{R} \mid C^\infty\text{-fun} \}$

" X " " complex m. ——— " — $\mathcal{O}_X(U) = \{ f : U \to \mathbb{C} \mid \text{holomor} \}$

Def: $f : U \longrightarrow \mathbb{C}$ is holomorphic if $f \circ \varphi_i^{-1} : \varphi_i(U_i \cap U) \longrightarrow \mathbb{C}$ is holomorphic

(or just one chart around any point)

$$\begin{tikzcd} U_i \cap U \arrow[r, "f|_{U_i \cap U}"] \arrow[d, "\varphi_i", "\cong"'] & \mathbb{C} \\ \varphi_i(U_i \cap U) \arrow[ur, "f \circ \varphi_i^{-1}"'] & \end{tikzcd}$$

$C^\infty_M(-)$ and $\mathcal{O}_X(-)$ are called sheaves of functions.

They are also sheaves of rings: for every $U \subseteq X$, we have a ring $\mathcal{O}_X(U)$ (addition and multiplication pointwise)

Moreover, for every $V \subseteq U \subseteq X$ open: the restriction map

$$\mathrm{res}^U_V = \rho^U_V = r_{U,V} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$$
$$f \longmapsto f|_V$$

satisfying $\rho^U_U = \mathrm{id}$ and if $W \subseteq V \subseteq U$: $\rho^V_W \circ \rho^U_V = \rho^U_W$

↑ def of presheaf of rings.

04/02/20

ex: M differentiable manifolds, C_M = sheaf of diff. fcn.