Complex Algebraic Geometry

Lecture #10: Blow-ups / bimeromorphic geometry
= birational

- Graphs, meromorphic maps
- Bimeromorphic maps and modifications
- Blow-ups: definition and properties
- Blow-ups: examples
- Famous theorems: res. of sing., res of indeterminacy locus, Chow lemmas, weak factorization

§1 Graphs & meromorphic maps

X, Y complex manifolds

$f : X \to Y$ holomorphic map $\Rightarrow$ graph $\Gamma_f = \{(x, f(x))\} \subset X \times Y$ is an analytic subset

(closed, locally given by vanishing of holo. fcns)

Rmk:
$$\begin{tikzcd} & \Gamma_f \arrow[dl, "\pi_1"', "\cong"] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Def: A meromorphic map $f : X \dashrightarrow Y$ is

(i) $U \subset X$ dense open

(ii) $f_U : U \to Y$ holomorphic

$f = (U, f_U)$

such that:

(iii) $\Gamma_f = \overline{\Gamma_{f_U}} \subset X \times Y$ is an analytic subset and $\Gamma_f \to X$ is proper (i.e. closed w/ compact fibers)

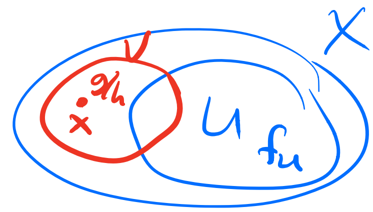
e.g. X, Y compact.

Exc (HW) Meromorphic fcns = meromorphic maps $X \dashrightarrow \mathbb{P}^1$

that is, $f_U : U \to \mathbb{C}$

$\forall x \in X$, $f_U|_{U \cap V} = (g/h)|_{U \cap V}$ for some

g, h holo on nbhd V of x.



Rmks:

- We identify two meromorphic maps (U, f_U) and (V, g_V) if $f_U = g_V$ on $U \cap V$.
- There $\exists$ maximal U s.th. $f = (U, f_U)$

  U is the largest open where f is a holomorphic map.

  $X \setminus U$ is the indeterminacy locus I_f.

Fact: The indeterminacy locus is an analytic subset.

Ex: $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}$ $\quad f(x,y) = \dfrac{x}{y}$.

Indeterminacy locus $Z(y)$. In this case, extends to
codim 1

$$\bar{f} : \mathbb{C}^2 \dashrightarrow \mathbb{P}^1 \quad f(x,y) = (x : y)$$

$$\mathbb{C} = \{(x:y) : y \neq 0\} = \left\{\left(\tfrac{x}{y} : 1\right)\right\} \subset \mathbb{P}^1$$

Indeterminacy locus of $\bar{f}$ is $Z(x,y)$. codim 2

Fact: X, Y cplx mflds, Y compact

$f : X \dashrightarrow Y$ meromorphic, $\Rightarrow I_f$ has codim ≥ 2.

Exc (HW) Composition of meromorphic maps is meromorphic.

Ex: (Meromorphic sections of line bundles)

Recall Weil divisor $D = \sum a_i [D_i]$ $\quad a_i \in \mathbb{Z}$, $D_i = Z(f_{ij})$ loc on U_j

$\Updownarrow$

(Lecture 7) Cartier div $D = \{(U_j, \prod f_{ij}^{a_i} = g_j)\}$ ← meromorphic fcn

Lecture 8?

$\rightsquigarrow$ line bundle $L = \mathcal{O}(D) =$

$\{X = \bigcup U_j, \underbrace{g_{ij}}_{\text{transition fcns for } L} = g_i/g_j$ non-van. holo fcn on $U_i \cap U_j\}$

Correspondence:

$$\left\{\begin{array}{l}\underbrace{\text{effective}}_{a_i \geq 0} \text{ divisors } D \\ \text{with } \mathcal{O}(D) = L\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{holo sections} \\ \text{of } L\end{array}\right\}$$

$L \overset{loc}{\cong} X \times \mathbb{C}$

$$D = \underbrace{Z(s)}_{\text{with mult.}} \longleftarrow\!\mapsto s \qquad \begin{array}{c} L \\ s \uparrow \downarrow \pi \\ X \end{array}$$

locally s is holo fcn up to mult with $\lambda \in \mathbb{C}^*$

$$\left\{\begin{array}{l}\text{divisors } D \\ \text{with } \mathcal{O}(D) = L\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{meromorphic sections } s \\ \text{of } L,\ s \not\equiv 0,\ s \not\equiv \infty\end{array}\right\}$$

$$= \{s : X \dashrightarrow \bar{L} : \pi \circ s = id_X,\ {}^{s \not\equiv 0}_{s \not\equiv \infty}\}$$

Here $\bar{L}$ is the projective line bundle corresponding to L, that is, locally $U_j \times \mathbb{P}^1$ with same transition fcns g_{ij}. $L \subset \bar{L}$ and $\bar{L} \setminus L = X \times \{\infty\}$

Exc: Compare with previous definition of meromorphic sections

Ex: Recall $K(\mathbb{P}^n)$ = rat'l fcns of $x_0, \dots, x_n$.

- $Z(x_0) \longleftrightarrow$ holo. section of $\mathcal{O}(1)$
- $Z\left(\frac{1}{x_0}\right) \longleftrightarrow$ mero. section of $\mathcal{O}(-1)$
- $Z\left(\frac{x_0 x_1 x_2^2}{x_3 x_4}\right) \longleftrightarrow$ mero. section of $\mathcal{O}(2)$
- $\frac{x_0 x_1 x_2^2}{x_3 x_4} : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ mero fcn.

indeterminacy loci are $Z(x_0 x_1 x_2, x_3 x_4)$

Ex: (linear systems) Lecture 9

$s_0, s_1, \dots, s_n \in \Gamma(X, L)$ = sections of L

$V = \mathrm{Span}\{s_0, \dots, s_n\} \subset \Gamma(X, L)$

gives (HW) $f : X \overset{|V|}{\dashrightarrow} \mathbb{P}^n$ meromorphic map

$x \longmapsto (s_0(x) : \dots : s_n(x))$

Indeterminacy locus $I_f = Z(s_0) \cap \dots \cap Z(s_n)$

$= Z(s_0, \dots, s_n)$

$\subseteq \underbrace{Bs(V)}_{\text{base locus}} = Bs(s_0, \dots, s_n)$

Rmk: $f: X \dashrightarrow Y$ meromorphic map gives

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, dashed, "\varphi\circ f"'] & & Y \arrow[dl, dashed, "\varphi"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

pullback of mero fcns $\quad f^*: K(Y) \longrightarrow K(X).$

$$\varphi \longmapsto \varphi\circ f$$

§ Meromorphic maps / fns in algebraic geometry

We have not given a precise definition of algebraic varieties but roughly $X = \bigcup U_i$ where $U_i = Z(\underbrace{f_{i1}, \dots, f_{ij}}_{\text{polynomials}}) \subset \mathbb{C}^n$ and transition functions b/w the U_i's are quotients of polynomials.

On X we have a sheaf of algebraic functions $\mathcal{O}_X$.
Algebraic functions on $U \subset X$ are those that locally are quotients of polynomials w/o poles over U.
So by definition, algebraic fcns on U give meromorphic fcn on X.

$$\underbrace{Rat(X)}_{\text{rational fcns}} := \{U \subset X,\ f \text{ alg fcn on } U\} \subset K(X) := \text{mero fcn on } X$$

If X compact, then $Rat(X) = K(X)$, but not o/w:

Ex: $e^{1/z}$ on $\mathbb{C} \setminus \{0\}$ does not give mero fcn on $\mathbb{C}$ (not algebraic)

Similarly, rational maps from X to Y are just $(U, f: U \xrightarrow{alg} Y)$ and these are always meromorphic, that is, cond (iii) in def'n of meromorphic map always satisfied.

§ Bimeromorphic maps

Def: $f : X \dashrightarrow Y$ is bimeromorphic if
$\Gamma_f \subset X \times Y \cong Y \times X$ also defines a meromorphic map
$$f^{-1} : Y \dashrightarrow X$$
that is: • $\Gamma_f \longrightarrow Y$ is an isomorphism over some dense open $V \subset Y$.
$$\rightsquigarrow \quad f_V^{-1} = \pi_1 \circ \pi_2^{-1}$$
• $\Gamma_f \longrightarrow Y$ is proper $\Rightarrow$ (f_V^{-1}, V) meromorphic

Rmk: $f : X \dashrightarrow Y$ meromorphic, $f_U : U \longrightarrow Y$.
$f(x) \in Y$ if $x \in U$.
$f(x) = \pi_2(\pi_1^{-1}(x)) \subset Y$ if $x \notin U$ multivalued fcn.

Rmk f bimero, $f^* : K(Y) \to K(X)$ isomorphism.
Fact: Converse also holds: given X, Y cpt mflds and (?) and an iso $K(Y) \cong K(X)$
it arises from some bimero map $f : X \to Y$.

Rmk: If $f : X \dashrightarrow Y$ bimeromorphic, then have

$$\begin{tikzcd} & \Gamma_f \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\ X \arrow[ur, dashed, "\pi_1^{-1}"] & & Y \arrow[ul, dashed, "\pi_2^{-1}"'] \end{tikzcd}$$

π_1, π_2 are proper holomorphic maps that are also bimerom,
that is, π_1, π_2 isomorphisms over some dense opens

Def: A modification is a map $f: X \longrightarrow Y$ which is
(i) proper
(ii) an isomorphism over some dense open $\mathcal{U}$.

$\iff$ a map $f: X \longrightarrow Y$ which is bimeromorphic.

Rmk above says that every bimero. $X \dashrightarrow Y$ factors as

$$\begin{tikzcd} & Z \arrow[dl, "\text{modif}"'] \arrow[dr, "\text{modif}"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

Warning: $Z = \Gamma_f$ need not be a manifold!
could be singular.

Thm (Resolution of singularities, Hironaka '64)
Every singular analytic subset Z can be desingularized:
- $\exists \underset{\text{mfld}}{\tilde{Z}} \longrightarrow Z$ modification

Stronger formulation:
- $\exists$ sequence of blow-ups $\underset{\text{mfld}}{\tilde{Z}} = Z_n \to \cdots \to Z_1 \to Z$

Even stronger:

Thm (Embedded res. of sing., Hironaka '64)

If $Z \subset \underset{\text{mfld}}{W}$ analytic subset (e.g. $\Gamma_f \subset X \times Y$)

then $\exists$ sequence of blow-ups with smooth centers

$$\begin{tikzcd}
\overset{\text{mfld}}{W_n} \arrow[r] & \cdots \arrow[r] & \overset{\text{mfld}}{W_1} \arrow[r] & W \\
Z_n \arrow[r] \arrow[u, phantom, "\cup"] & \cdots \arrow[r] & Z_1 \arrow[r] \arrow[u, phantom, "\cup"] & Z \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

strict transform   strict transform

with some work

Cor: (Resolution of indeterminacy locus) Given $f : \overset{\text{mfld}}{X} \dashrightarrow \overset{\text{mfld}}{Y}$

$$\exists \text{ modif } \begin{tikzcd}
& \overset{\text{mfld}}{Z} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "f"'] & & Y
\end{tikzcd}$$

- p seq. of blow-ups w/ smooth centers
- q holo. map (modif. if f bimero.)

($\operatorname{tr\,deg}_{\mathbb{C}} K(X) = \dim X$)

Cor: If X compact connected Moishezon manifold

then $\exists$ seq. of blow-ups $\tilde{X} \to \cdots \to X$ s.th. $\tilde{X}$ is projective ($\tilde{X} \subset \mathbb{P}^n$)

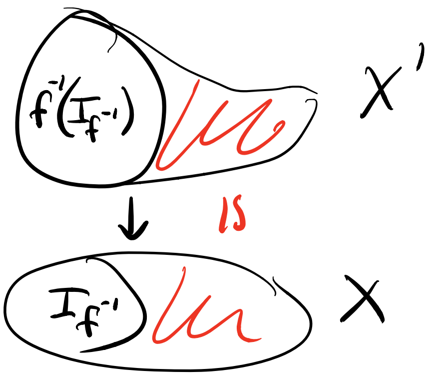
"pf":
- Pick $f : X \dashrightarrow Y \subset \mathbb{P}^n$ ($\exists$ by Moishezon)
- Resolve I_f.

§ Blow-ups (with smooth centers)

SLOGAN: Blow-ups are

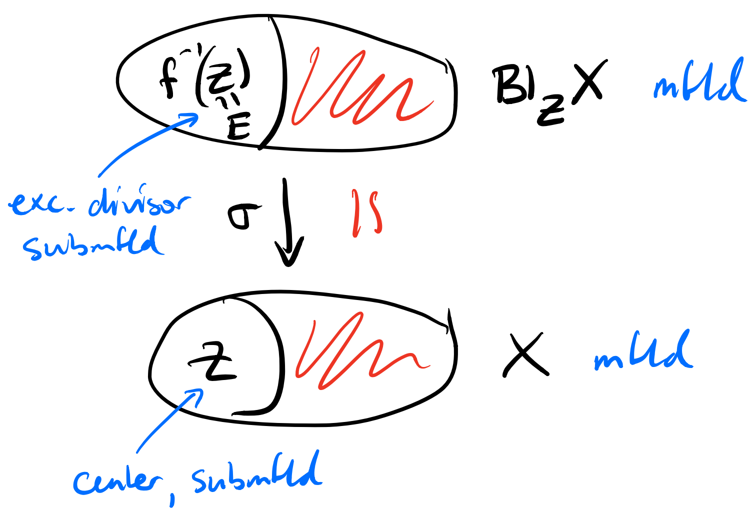
- very explicit modifications
- building blocks of modifications

Picture of general modification $f: X' \longrightarrow X$



"smooth center"

Picture of blow-up of X with center $Z \subset X$ closed submfld



Def: Exceptional divisor of blow-up is $E = \sigma^{-1}(Z)$.

It is a submanifold of codimension 1 $\Rightarrow$ divisor.

§ $Bl_0 \mathbb{C}^n$

A "toy case" but also local picture of general blow-up

$X = \mathbb{C}^n$, center $Z = \{(0,0,\dots,0)\} = Z(x_1, x_2, \dots, x_n)$

$$\begin{tikzcd} Bl_0 \mathbb{C}^n = \underbrace{\mathcal{O}(-1) \arrow[r, "\pi"] \arrow[d, "\sigma"'] \;\; \mathbb{P}^{n-1}}_{\text{line bundle on } \mathbb{P}^{n-1}} \\ \mathbb{C}^n \end{tikzcd}$$

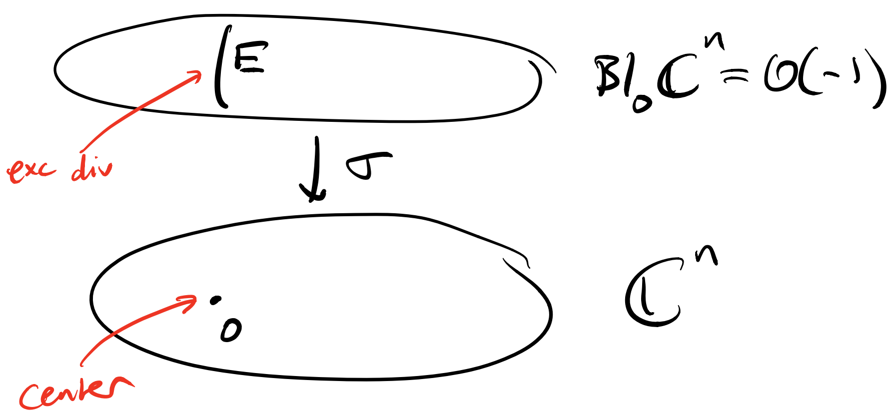
$x_1, \dots, x_n \quad z_1 : \dots : z_n$

<u>Recall</u>: $\mathcal{O}(-1) \subset \{(x, \ell) : x \in \ell\} \subset \mathbb{C}^n \times \underbrace{\mathbb{P}^{n-1}}_{= \text{lines through } 0 \text{ of } \mathbb{C}^n}$

- σ 1^{st} proj
- π 2^{nd} proj

$$\pi^{-1}(\ell) = \{x \in \mathbb{C}^n : x \in \ell\} \cong \mathbb{C}$$

$$\sigma^{-1}(x) = \begin{cases} \text{unique line through } x, & \text{if } x \neq 0 \\ \text{all lines} = \mathbb{P}^{n-1}, & \text{if } x = 0 \end{cases}$$



$E = \sigma^{-1}(0) = \mathbb{P}^{n-1}$ divisor on $Bl_0 \mathbb{C}^n = \mathcal{O}(-1)$

$\mathcal{O}(-1) = \{(x, z) : x_i z_j = x_j z_i \ \forall i, j\}$

$E = Z(x_1, \ldots, x_n)$ divisor ?!

Should be locally cut out by one holo. fcn

Locally on $\mathbb{P}^{n-1}$

Chart $\underbrace{z_1 \neq 0 \text{ on } \mathbb{P}^{n-1}}_{\cong \mathbb{C}^{n-1}} \rightsquigarrow$ chart $\underbrace{z_1 \neq 0 \text{ on } \mathbb{C}^n \times \mathbb{P}^{n-1}}_{\cong \mathbb{C}^n \times \mathbb{C}^{n-1}}$

Equations for $\mathcal{O}(-1)$ on this chart becomes:

$$\boxed{x_i = \frac{z_i}{z_1} x_1}$$

$\Rightarrow E = Z(x_1, \ldots, x_n) = Z(x_1)$ is a divisor!