Complex Algebraic Geometry

Lecture #11: Blow-ups

- Projection from a point
- Blow-ups: properties
- Blow-ups: examples

§ Projection from a point

$z_i = x_i / x_0$

$X = \mathbb{P}^n$, $\quad P = O = (\underset{x_0}{1} : \underset{x_1}{0} : \cdots : \underset{x_n}{0})$ $\qquad \mathbb{C}^n \subset \mathbb{P}^n$, $\quad \mathbb{C}^n = \{x_0 \neq 0\}$

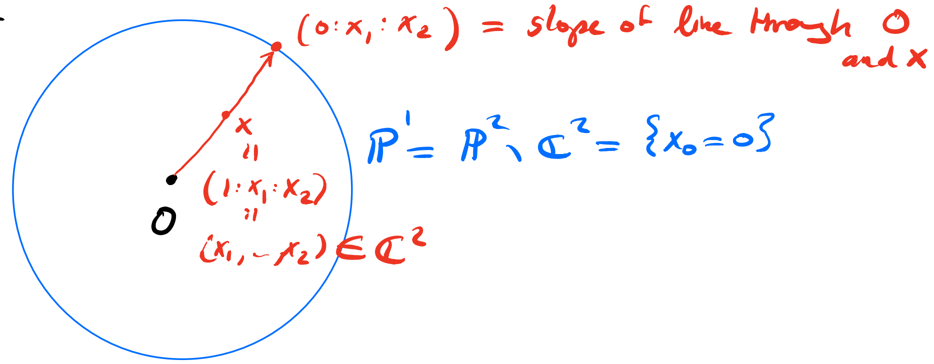
$V = \{x_1, \dots, x_n\} \subset \Gamma(X, \mathcal{O}(1))$ linear system, $\quad Bs(V) = O$

$$X \setminus Bs(V) \xrightarrow{|V|} \mathbb{P}(V) = \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$$

$$(x_0 : \cdots : x_n) \longmapsto (x_1 : \cdots : x_n) \longmapsto (0 : x_1 : \cdots : x_n)$$

Picture: $n = 2$

$\mathbb{P}^2$



$|V|$ is not defined at O.

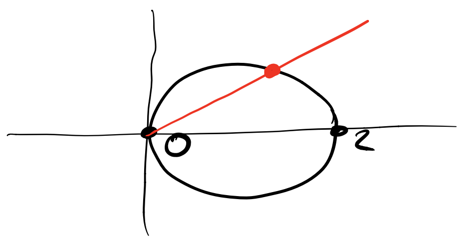
To get a map, replace $\mathbb{C}^2$ with $\{(x, \alpha) \text{ s.th. } x \text{ on line w/ slope } \alpha\} = Bl_O \mathbb{C}^2$

$$\begin{tikzcd}
\mathbb{C}^2 \setminus 0 \arrow[r, phantom, "\subset"] \arrow[d, equal] \arrow[rr, bend left, "|v|"] & Bl_0\, \mathbb{C}^2 \arrow[r] \arrow[d] & \mathbb{P}^1 \\
\mathbb{C}^2 \setminus 0 \arrow[r, phantom, "\subset"] & \mathbb{C}^2 &
\end{tikzcd}$$

__Ex 2:__ $X \subset \mathbb{P}^2$, X curve, project from 0.

$$X = Z\left((x-z)^2 + y^2 - z^2\right) \quad (\cong \mathbb{P}^1)$$

$$(x-1)^2 + y^2 = 1^2$$



$$X \setminus 0 \subset \mathbb{P}^2 \setminus 0 \longrightarrow \mathbb{P}^1$$

extends? $X \longrightarrow \mathbb{P}^1$?

$0 \longmapsto$ tangent slope at 0.

$$\begin{tikzcd}
\tilde{X} \arrow[r, phantom, "\subset"] \arrow[d, "\cong"'] & Bl_0\, \mathbb{P}^2 \arrow[r] \arrow[d, "\sigma"] & \mathbb{P}^1 \\
X \arrow[r, phantom, "\subset"] & \mathbb{P}^2 &
\end{tikzcd}$$

center

__Def:__ If $Bl_Z Y \xrightarrow{\sigma} Y$ blowup of Y in Z and $W \subset Y$ analytic set, then the strict transform of W is

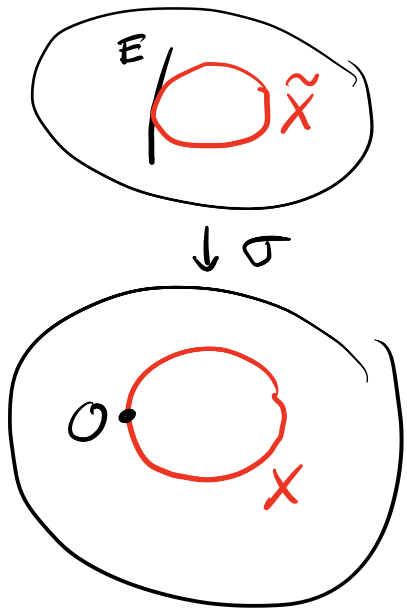
$$\tilde{W} = \overline{\sigma^{-1}(W \setminus Z)} = \overline{\sigma^{-1}(W) \setminus E}$$

The total transform of W is $\sigma^{-1}(W)$.

<u>**Fact:**</u> $\tilde{W} \subset Bl_Z Y$ is an analytic set.
([Huybrechts, Ch 1], local theory, irreducible components)

<u>Back to example:</u> $\qquad E$ exc div $\cong \mathbb{P}^1$

$$\sigma^{-1}(X) = Z\left((x-z)^2 + y^2 - z^2\right) \qquad Bl_0 \mathbb{P}^2$$



$$X = Z\left((x-z)^2 + y^2 - z^2\right) \qquad \mathbb{P}^2$$

<u>Locally:</u>

(1) $\mathbb{C}^2 \subset \mathbb{P}^2 \qquad X = Z\left((x-1)^2 + y^2 - 1\right)$

(2) $\mathbb{C}^2 \overset{s \neq 0}{\subset} Bl_0 \mathbb{C}^2 \subset \mathbb{C}^2 \times \mathbb{P}^1$

$\quad \mathbb{C}^2 \overset{t \neq 0}{\subset} \mathcal{O}(-1) = \{(x,y),(s:t) \,:\, xt = ys\}$

"(x,y) on line $(s:t)$"

Chart $s \neq 0$: (all slopes but vertical lines)

$$\mathbb{C}^2 = \{s \neq 0\} = Bl_0\,\mathbb{C}^2 \cap \underbrace{\mathbb{C}^1}_{s \neq 0}$$

coord t/s

$$= \{ (x,y),\ (t/s) \ : \ x\, t/s = y \}$$

$$= \{ (x, t/s) \in \mathbb{C}^2 \}$$

$$\sigma^{-1}(X) \text{ on this chart} = Z\left((x-1)^2 + (x\, t/s)^2 - 1 \right)$$

$$= Z\Big(\underset{\uparrow \text{ exc div}}{x} \underbrace{\left(x\left(1 + (t/s)^2\right) - 2 \right)}_{\text{strict thm}} \Big)$$

exc div, strict thm are disjoint

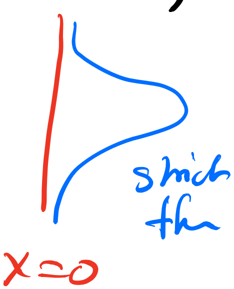


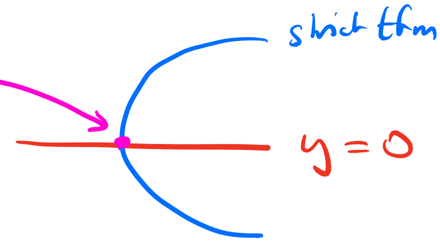
Chart $t \neq 0$

$$\mathbb{C}^2 = \{ (y, s/t) \} \qquad x = y\, s/t$$

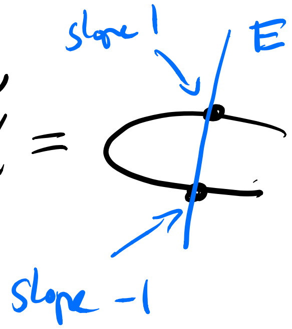
$$\sigma^{-1}(X) = Z\left((y\, s/t - 1)^2 + y^2 - 1 \right)$$

$$= Z\Big(\underset{\uparrow \text{ exc div}}{y} \underbrace{\left(y\left((s/t)^2 + 1\right) - 2 s/t \right)}_{\text{strict thm}} \Big)$$

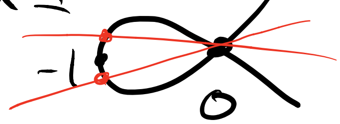
intersects at $y=0$, $s/t = 0$



Ex 3: Replace smooth conic X with a singular cubic.

$\tilde{X} =$ 

$$\begin{tikzcd} \tilde{X} \subset Bl_0\mathbb{P}^2 \arrow[r] \arrow[d, "\sigma"] & \mathbb{P}^1 \\ X \subset \mathbb{P}^2 \arrow[ur, dashed, "\text{projection from } 0"'] & \end{tikzcd}$$

$X =$ 

$\subset \mathbb{P}^2$

$= z(y^2z - x^2(x+z))$

0 has two different limit slopes.

Rmk: $X \setminus 0 \to \mathbb{P}^1$ injective b/c X cubic and singular at $0 \Rightarrow$ all lines through 0 meets X w/ mult $= 2$.

Exc: Verify picture above.

§ Definition of blow-up

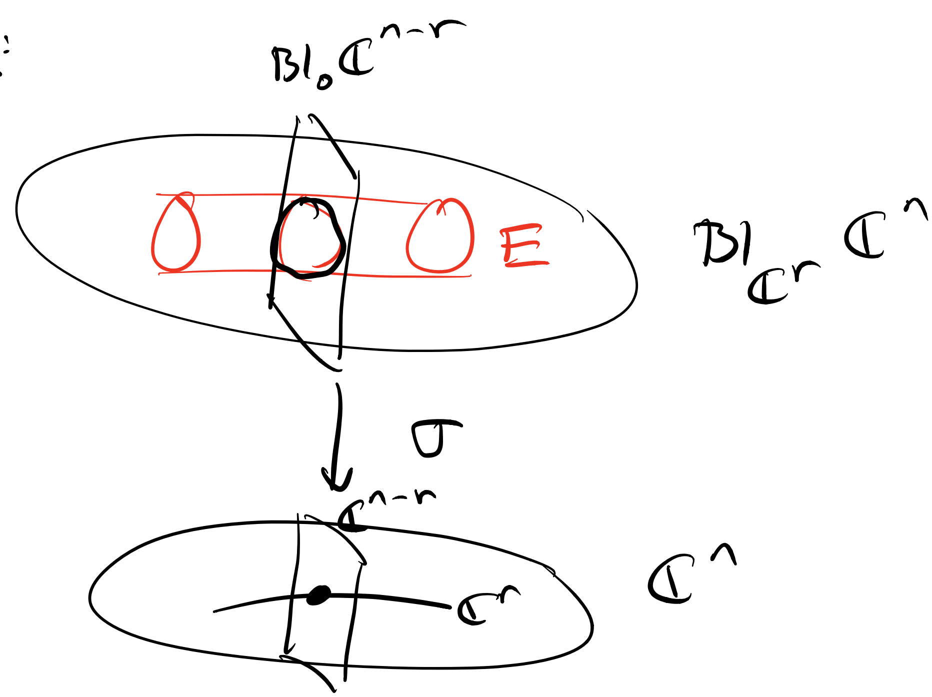
Previous lecture: $Bl_0 \mathbb{C}^n := \mathcal{O}(-1)$

Goal: $Bl_Z X$, X cpk mfld, $Z \subset X$ closed submfld

First step: Define $Bl_{\mathbb{C}^r} \mathbb{C}^n$, $\mathbb{C}^r = \{(x_1, \dots, x_r, 0, 0, \dots, 0)\}$

$$E = \sigma^{-1}(\mathbb{C}^r) = \mathbb{P}(N_{\mathbb{C}^r/\mathbb{C}^n})$$
$$\downarrow \quad \leftarrow \quad \mathbb{P}^{n-r-1}\text{-bundle}$$
$$\mathbb{C}^r$$

Picture:



Def: $\mathrm{Bl}_{\mathbb{C}^r}\mathbb{C}^n := \mathbb{C}^r \times \mathrm{Bl}_0\mathbb{C}^{n-r} \subset \mathbb{C}^n \times \mathbb{P}^{n-r+1}$

$$= \left\{(x_1, x_2, \dots, x_n), (y_{r+1} : \dots : y_n) : \begin{array}{c} x_i y_j = x_j y_i \\ \forall i,j > r \end{array}\right\}$$

Manifold with $n-r$ charts $(y_j \neq 0)$ each isomorphic to $\mathbb{C}^n$.

Second step: For general $Z \subset X$, pick local charts

$$\begin{array}{ccc} Z & \subset & X \\ \cup & & \cup \\ Z \cap U = \mathbb{C}^r \cap U & \subset & U \\ \cap & & \cap \\ \mathbb{C}^r & \subset & \mathbb{C}^n. \end{array}$$

and prove that this glues.

Independent of choice of charts?

Somewhat painful

§ Properties of blow-ups

Proposition: Let $Z \subset X$ be closed submanifold

(i) $\sigma^{-1}(Z)$ is a divisor.

(ii) The blow-up is universal w/ property (i), that is

$\forall\, Y \xrightarrow{f} X$, Y manifold, $f^{-1}(Z)$ divisor

$$\Longrightarrow \quad \begin{tikzcd} & \exists!\ \mathrm{Bl}_Z X \arrow[d, "\sigma"] \\ Y \arrow[ur, dashed] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\circ"] & X \end{tikzcd}$$

Proving that $\mathrm{Bl}_{\mathbb{C}^r} \mathbb{C}^n$ has universal property

$\Longrightarrow$ gluing for free.

Rmk: $\sigma : \mathrm{Bl}_Z X \longrightarrow X$ is:

$\begin{cases} \text{(i) proper (locally } \mathrm{Bl}_Z X \overset{\text{closed}}{\subset} X \times \mathbb{P}^{n-r-1}) \\ \text{(ii) an isomorphism over } X - Z. \end{cases}$

$\Longleftrightarrow$ σ is a modification.

Last time: Mentioned Hironaka $\Longrightarrow$ modifications are dominated by blow-ups

$$\begin{tikzcd} & \exists\ Z \arrow[dl, "\text{blowup}"'] \arrow[ddl, bend left, "\text{blow-up}"] \\ Y \arrow[d, "\forall \text{ modification}"'] & \\ X & \end{tikzcd}$$

§ Resolution of singularities (plane curves)

Def: $Z \subset \mathbb{C}^n$ divisor. The order of Z at $P \in \mathbb{C}^n$ is the smallest degree of a monomial appearing in the Taylor series expansion at P.

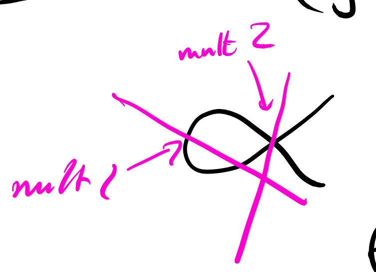
Locally: $Z = Z(f)$ f holo fcn. Locally at P

$$f = \sum a_\alpha z^\alpha \qquad \mathrm{ord}_Z(P) = \min\{|\alpha| : a_\alpha \neq 0\}$$

Rmk: $\mathrm{ord}_Z(P) > 0 \iff f(P) = 0 \iff P \in Z$.

$\mathrm{ord}_Z(P) = 1 \iff Z$ submfld at P.

Ex: $Z = Z\big(y^2 - x^2(x-1)\big) \subset \mathbb{C}^2$



$$\mathrm{ord}_Z(0) = 2$$

$$f = \underbrace{y^2 + x^2}_{\deg 2} - x^3$$

$\mathrm{ord}_Z(P) = 1$ at $Z \setminus \{0\}$.

Rmk: $Z \subset \mathbb{C}^2$. $\mathrm{ord}_Z(P) = \left\{ \begin{array}{l} \text{mult at } P \text{ of } f \text{ on } L \text{ for} \\ \text{a general line through } P \end{array} \right\}$

$\mathrm{ord}_Z(P) \geq n \iff (n-1)^{th}$ partials of f vanishes at P

$$\underbrace{\{ P : \mathrm{ord}_Z(P) \geq n \}}_{\text{analytic set}} = Z\left(\frac{\partial^\alpha f}{\partial x^\alpha} : |\alpha| \leq n-1 \right)$$

Thm (Hironaka) $Z \subset X$ analytic subset of mfld X
$\exists$ seq blow-ups $\tilde{X} \to X$ s.th. strict tfm $\tilde{Z}$ is smooth

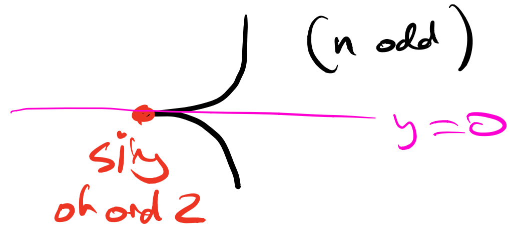
Resolution algorithm of $Z \subset X$

- X 2-dim'l manifold
- Z divisor, generically smooth $\left(\underbrace{Z_{sing}}_{\{\mathrm{ord}_P Z \geq 2\}} \neq Z \right)$

Algo: Blow-up any point with order > 1.

Lemma: $\mathrm{ord}_Z(P) = d > 1$, then after a finite number of blow-ups $\mathrm{ord}_{\tilde{Z}}(Q) < d$ $\forall Q$ above P.

Ex: $Z = Z(y^2 - x^n)$ (n odd)



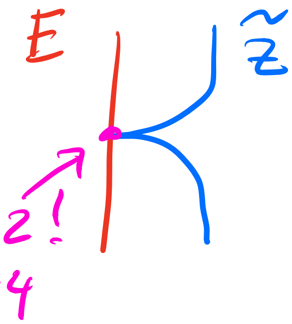
Blow-up 0

Look at chart $(s:t)$, $s \neq 0$

$$xt = ys$$

$$y = x\,t/s$$

$$\begin{aligned} f &= y^2 - x^n \\ &= (x\,t/s)^2 - x^n \\ &= \underbrace{x^2}_{\text{2 copies of the exc div}}\underbrace{\left((t/s)^2 - x^{n-2}\right)}_{\text{strict tfm}} \end{aligned}$$



After $\lfloor n/2 \rfloor$ blow-ups order drops to 1.