Complex Algebraic Geometry

Lecture #12: Projectivity

- Comments on proper maps
- Comments on analytic sets
- Comments on meromorphic maps
- Complex projective varieties
- Kodaira embedding theorem (statement)

Proper maps

proper = relative notion of compactness: $f: X \to Y$ proper $\Leftrightarrow$ $\boxed{f^{-1}(K) \text{ compact } \forall K \subset Y \text{ compact}}$ HW (0a)

From def: X compact $\Rightarrow$ $f: X \to Y$ proper.

Ex: Y compact, then $X \times Y \xrightarrow{\pi_1} X$ proper.

Ex: $f: X \to Y$ proper, $Z \subset X$ closed subset, $f|_Z$ proper.

Exc: In def of $f: X \dashrightarrow Y$ meromorphic

$$\Gamma_f = \overline{\Gamma_{f_U}} \overset{\text{closed}}{\subset} X \times Y \text{ proper over } X \quad \text{automatic if } Y \text{ compact}$$

Slogan: Proper maps have various "finiteness" properties.

Thm (Remmert, proper mapping thm) X, Y cplx mflds

$f: X \to Y$ proper, $Z \subset X$ analytic, $f(Z) \subset Y$ analytic.

($\exists$ generalization of proper mapping thm to coherent sheaves
$R^i_* f(\mathcal{F})$ coherent if $\mathcal{F}$ coherent)

Analytic sets

Recall: $Z \subset X$ analytic if locally on X, $Z = Z(f_1, \dots, f_n)$ where f_i holo.

Rmk: Finite unions & intersections of analytic sets are analytic.

b/c if $Z = Z(f_1, \dots, f_n)$, $W = Z(g_1, \dots, g_m)$

$Z \cap W = Z(f_1, \dots, f_n, g_1, \dots, g_m)$

$Z \cup W = Z(f_i g_j : 1 \le i \le n, 1 \le j \le m)$

Fact: ("noetherian") descending chains of an. sets stabilize

$$Z_1 \supset Z_2 \supset Z_3 \supset \cdots \quad \Rightarrow \quad \exists n \quad Z_n = Z_{n+1} = \cdots$$

$\Rightarrow$ infinite intersections of an. sets are analytic.

Def: An analytic subset Z is irreducible if Z is not a union of two properly contained analytic subsets.

Def: Let $Z^* \subset Z$ be largest open where Z is a manifold. (poss diff dim's on conn. cpts)

Fact:
- $Z^* \subset Z$ dense open.
- If $Z_i^* \subset Z^*$ is a connected component, then $\overline{Z_i^*}$ is an $\underbrace{\text{irreducible}}_{\text{"obvious"}}$ $\underbrace{\text{analytic}}_{\text{not obvious}}$ subset

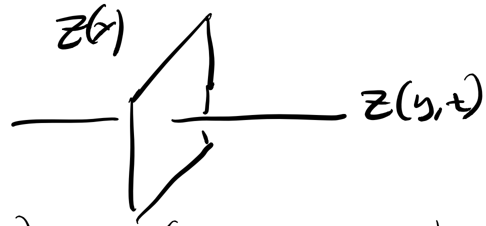
$\Rightarrow$ • Z is a (loc. fin.) union of irreducible an. sets

Def: $\dim Z := \dim Z^* = \max_i (\dim Z_i^*)$

Rmk: If X connected mfld, then $\dim Z < \dim X$ unless $Z = X$.

Ex: $X = \mathbb{C}^3$.

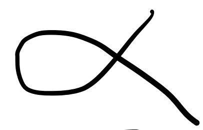
(a) $Z = \underbrace{Z(x)}_{\text{irr}} \cup \underbrace{Z(y,z)}_{\text{irr}}$



$$Z^* = Z \setminus \{0\} = \underbrace{(Z(x) \setminus \{0\})}_{\text{mfld dim } 2} \amalg \underbrace{(Z(y,z) \setminus \{0\})}_{\text{mfld of dim } 1}$$

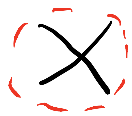
(b) irreducible $\not\Rightarrow$ locally irreducible

$Z = Z(y^2 - x^2(x+1))$



irreducible

$U = D(0, \tfrac{1}{2})$



Choose $\sqrt{x+1}$ on U.

$$Z \cap U = Z(y - x\sqrt{x+1}) \cup Z(y + x\sqrt{x+1})$$

Lemma: If Z, W are analytic, then $\overline{Z \setminus W}$ analytic

sketch of pf: Look at irreducible components.

If Z irreducible, either $Z \subseteq W \Rightarrow \overline{Z \setminus W} = \emptyset$

or $\underset{\text{smaller dim}}{Z \cap W \subset Z} \Rightarrow \overline{Z \setminus W} = Z$.

$\square$

Can enlarge category of complex mflds to possibly singular complex spaces and treat an. sets as subspaces.

Meromorphic maps

$$\begin{tikzcd} & Z \arrow[dl, "\text{holo}"'] \arrow[dr, "\text{holo}"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

Ex: $f: \mathbb{C} \setminus 0 \to \mathbb{C}$

$z \longmapsto e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$

not meromorphic fcn.

$\Gamma_f \subset (\mathbb{C} \setminus 0) \times \mathbb{C}$ $\quad \Gamma_f = \{(z,w) : w = e^{1/z}\}$
$= Z(w - e^{1/z})$

$z \rightsquigarrow 0 \Rightarrow w \rightsquigarrow$ all values of $\mathbb{C}$

$\overline{\Gamma_f} \subset \mathbb{C} \times \mathbb{C}$ $\quad \overline{\Gamma_f} = \underset{z \neq 0}{Z(w - e^{1/z})} \cup (0 \times \mathbb{C})$

$\subset \mathbb{C} \times \mathbb{P}^1$ $\quad \cup (0 \times \mathbb{P}^1)$

closed but not analytic $\quad$ (proper/$\mathbb{C}$)

Ex: $f: \mathbb{C} \times \mathbb{C} \dashrightarrow \mathbb{P}^1$

$x, y \longmapsto (x : y)$

$y = kx, \; x \rightsquigarrow 0 \Rightarrow (x : y) \rightsquigarrow (1 : k)$
$x = hy \quad y \rightsquigarrow 0 \qquad\qquad (h : 1)$

$\Gamma_f \subset \mathbb{C}^2 \times \mathbb{P}^1$ $\quad \Gamma_f = \{((x,y),(s:t)) : \begin{vmatrix} x & y \\ s & t \end{vmatrix} = 0\}$

$f(z)$ mero at 0, $\quad f(z) = z^{-d} \underbrace{g(z)}$ holo

$y = \frac{1}{z}$ grows slower than $e^{1/z}$ when $z \to 0$

$f(z_1, \ldots, z_n)$ mero $\quad f(z) = g(z)/h(z)$ loc

§ Complex projective varieties

<u>Def</u>: A complex proj variety $Z \subset \mathbb{P}^n$ is the vanishing locus of some homogeneous polynomials $f_1, \dots, f_r$

not fcns on $\mathbb{P}^n$
but $Z(f_1, \dots, f_r)$
makes sense.

<u>Rmk</u>: Finite unions & intersections of cplx proj var's are such.
Infinite intersections as well
(use that $\mathbb{C}[x_0, \dots, x_n]$ is noetherian)

<u>Def</u>: The Zariski topology on $\mathbb{P}^n$ is the topology where cplx proj varieties are closed.

<u>Def</u>: If $Z \subset \mathbb{P}^n$ cplx proj variety, then Zariski topology on Z is the induced one, that is,
$W \subset Z$ closed $\iff$ $\exists$ cplx proj variety W' s.th. $W = Z \cap W'$.

<u>Rmk</u>: If $U \subset Z$ Zariski-open $\Rightarrow$ open in Euclidean top
(usual top on $\mathbb{P}^n$ restricted to Z

<u>Rmk</u>: cplx proj var's are analytic sets

Thm (Chow): On $\mathbb{P}^n$ every an. set is a cplx proj variety.

Def: A projective complex mfld is a mfld X that is a closed submanifold of $\mathbb{P}^n$ for some n.

Chow $\Rightarrow$ proj. cplx mflds are smooth cplx proj varieties.

Thm (Serre, GAGA) Every (holomorphic) vector bundle on a proj cplx mfld ($\Leftrightarrow$ smooth cplx proj variety) is algebraic.

Def: Let X be cplx proj variety. A holo v.b. E on X is algebraic if:

(i) $\exists X = \bigcup U_i$ s.th. $E \cong U_i \times \mathbb{C}^r$ and $U_i \subset X$ Zariski-open

(ii) transitions fcns $g_{ij}: U_i \cap U_j \longrightarrow GL(\mathbb{C}, r)$ are algebraic

= entries are alg fcns.

Def: Let $X \subset \mathbb{P}^n$ cplx proj variety, $U \subset X$ Zariski-open.

An alg. fcn on U is $f: U \xrightarrow{\text{holo}} \mathbb{C}$ s.th. locally

$$f(z) = \frac{g(z)}{h(z)}$$ g, h homogeneous polynomials of same degree.

§ Kodaira embedding theorem

Qstn: When is a compact cplx mfld projective?

Thm (Kodaira embedding theorem) (positive $\Leftrightarrow$ ample)
A compact cplx mfld X is proj $\Leftrightarrow$ $\exists$ positive line bundle L.

More precisely: If L is positive, then the linear system:

$$X \xrightarrow{|L^{\otimes r}|} \mathbb{P}(V) \qquad V = H^0(X, L^{\otimes r})$$

gives an embedding of X for all suff. large r.

"Def" L is positive if $\exists$ Hermitian metric on L with positive curvature.

Rmk: Conversely, if $X \subset \mathbb{P}^n$, let $L = \mathcal{O}(1)\big|_X$.

Then $V = H^0(X, L)$ gives an embedding of X in $\mathbb{P}(V) \cong \mathbb{P}^m$ (typically $\mathbb{P}(V) = \mathbb{P}^n$)

$\mathcal{O}(1)$ on $\mathbb{P}^n$ is positive (Fubini–Study metric on $\mathcal{O}(1)$)

§ When does a line bundle L give an embedding $X \subset \mathbb{P}^n$?

(= when is L very ample ?)

(1) Need that $X \overset{\varphi}{-\,-\rightarrow} \mathbb{P}(V)$ is def. everywhere.

(2) Need that φ is injective.

(3) Need that image of φ is a manifold

$\iff$ φ is injective on tangent spaces.

(1) $V = H^0(X, L)$ $\qquad X \dashrightarrow \mathbb{P}(V) \cong \mathbb{P}^n$

basis $s_0, \dots, s_n$ $\qquad x \longmapsto (s_0(x) : \dots : s_n(x))$

$s_0, \dots, s_n$ should vanish simultaneously, i.e., $\forall x \ \exists i \ s_i(x) \neq 0$.

$\iff$ $H^0(X, L) \longrightarrow L(x) = \pi^{-1}(x)$ surjective $\forall x$.

$$\begin{gathered} \| \\ \left\{ \begin{tikzcd} L \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd} \right\} \end{gathered}$$

(b/c if $s_i(x) \neq 0$, $(\lambda s_i)(x) = \lambda\, s_i(x)$)

(2) similarly this amounts to

$$H^0(X, L) \longrightarrow L(x) \times L(y) \quad \text{surjective} \quad \forall x \neq y$$

(3) related to (2) when $y \rightsquigarrow x$.

Prove this using blow-ups & Serre/ Kodaira vanishing