Complex Algebraic Geometry

Lecture #13: Hermitian metrics, Lefschetz theorems

- Lefschetz theorems & Kodaira vanishing for projective manifolds
  (Hard Lefschetz & Lefschetz hyperplane theorem)
- Hermitian metrics
- Fubini-Study metric on $\mathbb{P}^n$
- Kähler manifolds
- Hermitian line bundles and positivity
- Kodaira vanishing and Serre vanishing

§ Lefschetz theorems

M oriented compact manifold of dim n

<u>Poincaré duality</u>: $H^k(M,\mathbb{R}) \xrightarrow{\cong} H^{n-k}(M,\mathbb{R})^*$

$$\omega \longmapsto \int_M \omega \wedge (-)$$

X complex n-dim projective manifold (real $2n$-dim)

Poincaré duality $\quad H^{n+k}(M,\mathbb{C}) \xrightarrow{\cong} H^{n-k}(M,\mathbb{C})^*$

$$X \hookrightarrow \mathbb{P}^N$$
$$\cup$$
H hyperplane (divisor) $\rightsquigarrow$ line bundle $\mathcal{O}(H) = \mathcal{O}(1)$

<u>Ex</u>: $H = \{z_0 = 0\}$

$D = X \cap H$ is a (smooth) divisor for generic H.
$L := \mathcal{O}(D) = \mathcal{O}(1)|_X$ ample line bundle, that is,

$s_0, s_1, \dots, s_N$ sections of L gives $X \xrightarrow{|L|} \mathbb{P}^N$

$$x \longmapsto (s_0(x) : \dots : s_N(x))$$

Recall first Chern class $c_1(L) \in H^2(X,\mathbb{Z})$:
Exponential sequence: (an exact sequence of sheaves)

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

Gives
$$\begin{array}{ccc} \mathrm{Pic}\,X = H^1(X, \mathcal{O}_X^\times) & \xrightarrow{\ \partial\ } & H^2(X, \mathbb{Z}) \\ \rotatebox{90}{$\in$} & & \\ L & \longmapsto & c_1(L) \end{array}$$

<u>Thm</u> (Hard Lefschetz) X proj, cplx mfld, $L = \mathcal{O}(1)$.

$$c_1(L)^k \cap - \; : \; H^{n-k}(X, \mathbb{C}) \longrightarrow H^{n+k}(X, \mathbb{C})$$

isomorphism $\forall k = 0, 1, 2, \ldots, n$.

"Explains" Poincaré duality more explicitly.

<u>Thm</u> (Weak Lefschetz) X proj, cplx mfld
$D = H \cap X$, $L = \mathcal{O}(D) = \mathcal{O}(1)$ as above:

$$H^k(X, \mathbb{C}) \longrightarrow H^k(D, \mathbb{C})$$

is
- injective $\forall k \leq n-1$
- bijective $\forall k \leq n-2$.

<u>Rmk</u>: Hard Lefschetz $\Rightarrow$ injectivity part of weak Lefschetz

$$c_1(L) \cap - : H^k(X, \mathbb{C}) \longrightarrow H^k(D, \mathbb{C}) \longrightarrow H^{k+2}(X, \mathbb{C})$$

Using Lefschetz hyperplane theorem one fairly easily proves:

Thm (Kodaira vanishing) X projective cplx mfld, $L = \mathcal{O}(1)$.

$$H^q(X, \underbrace{K_X} \otimes L) = 0 \qquad \forall q > 0.$$

K_X = canonical line bundle $= \det(\underbrace{\Omega_X})$ cotangent bundle $(TX)^*$

see e.g. Lazarsfeld "Positivity in algebraic geometry"

Recall: Dolbeault cohomology. Let $\Omega_X^p = \Lambda^p \Omega_X$

$$H^{p,q}(X) := H^q(X, \Omega_X^p) = H_{\bar\partial}^q(X, \Omega_X^p)$$
$$= \{\bar\partial\text{-closed } (p,q)\text{-forms}\} / \bar\partial((p,q-1)\text{-forms})$$
$$= \ker(\bar\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}) / \operatorname{im}(\bar\partial : \mathcal{A}^{p,q-1} \longrightarrow \mathcal{A}^{p,q})$$

Like de Rham cohomology but $\bar\partial$ instead of $d = \partial + \bar\partial$

Thm: X projective cplx mfld:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

In particular: $c_1(L) \in H^{1,1}(X) \subset H^2(X, \mathbb{C})$ so can represent $c_1(L)$ by a closed $(1,1)$-form ω, locally:

$$\omega = \sum h_{ij} \, dz_i \wedge d\bar z_j$$

§ Hermitian metrics

Def: A hermitian metric h on a complex vector bundle E on X is a positive definite hermitian form $h_p(-,-)$ on E_p $\forall p \in X$ such that h_p varies smoothly with p.

(differentiably, not holo!)

sesqui-linear & conj. symm

Rmk: $h_p(ix, iy) = h_p(x,y)$ (h compatible w/ cplx structure)

Def: $g_p(-,-) := \operatorname{Re} h_p(-,-)$

g_p is a scalar product on E_p (forget complex structure)
and $g_p(ix,iy) = g_p(x,y)$ (g compatible w/ cplx structure)

$$g(x, iy) = \operatorname{Re}\big(h(x, iy)\big) = \operatorname{Re}\big(-i h(x,y)\big) = \operatorname{Im} h(x,y)$$

$$\Rightarrow h(x,y) = g(x,y) + i g(x, iy)$$

So scalar product comp. w/ cplx structure g
$\overset{1-1}{\longleftrightarrow}$ hermitian metric h

Def: The fundamental form ω of hermitian metric h is:

$$\boxed{\omega = -\operatorname{Im}(h)}$$

ω is an anti-symmetric, real bilinear form on E such that
$\omega(ix, iy) = \omega(x,y)$ (compatible w/ cplx structure)
and ω is positive: $\omega(x, ix) = -\operatorname{Im}(h(x,ix)) = -\operatorname{Im}(-ih(x,x))$
$= g(x,x) > 0$

<u>Rmk</u>: g and h are determined by ω.

- $g(x,y) = \omega(x, iy)$
- $h(x,y) = g(x,y) - i\omega(x,y)$

<u>Fact</u>: Every cplx vector bundle admits a hermitian metric.

<u>Def</u>: A hermitian structure on X is a hermitian metric on TX.

(<u>Recall</u>: A Riemann mfld M is a manifold M with a <u>Riemannian metric</u> on TM, i.e., scalar products on T_pM.)

<u>Rmk</u>: h on $TX \rightsquigarrow g$ on $TX \rightsquigarrow (X,g)$ Riemann mfld.

<u>Conversely</u>: If (X,g) Riemann complex mfld and $g(ix,iy) = g(x,y) \rightsquigarrow$ hermitian structure h on X

The fundamental form of a hermitian structure is a $(1,1)$-form

Locally:

$$\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$$

ω is **real**, $\omega(x,y) \in \mathbb{R}$, $\forall x,y \in T_pX \iff \bar{\omega} = \omega$.

ω is **positive**, $\omega(x,ix) \in \mathbb{R}_+ \iff (h_{ij})$ positive definite.
$\forall x \in T_pX$ complex matrix

§ Fubini-Study metric on $\mathbb{P}^n$ (3.1.9 (i))

Will describe fundamental form of Fubini-Study metric on $\mathbb{P}^n$:

ω_{FS} a real positive (1,1)-form

Std cover: $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$, $U_i = \{(z_0 : z_1 : \cdots : z_n) : z_i \neq 0\}$

$$U_i \cong \mathbb{C}^n = \left\{\left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right)\right\} = \{(w_1, \ldots, w_n)\}$$

$$\omega_{FS}\big|_{U_i} = \omega_i := \frac{i}{2\pi}\partial\bar\partial \log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_i}\right|^2\right)$$

$$= \frac{i}{2\pi}\partial\bar\partial \log\left(\sum_{i=1}^{n}|w_i|^2 + 1\right)$$

also $l=i$!

Verify that the ω_i glue. (+ real, positive)

Rmk: ω_{FS} is closed: $d\omega_{FS} = 0$

$$\Rightarrow [\omega_{FS}] \in H^{1,1}(\mathbb{P}^n)$$

Fact: $[\omega_{FS}] = c_1(\mathcal{O}(1))$

§ Kähler manifolds

Def: A complex mfld is Kähler if $\exists$ a closed, real, positive $(1,1)$–form (i.e., a hermitian structure h s.th. $d\omega = 0$)

Ex: ω_{FS} makes $\mathbb{P}^n$ Kähler

$\Rightarrow$ every proj. manifold is Kähler

Many results for proj. mflds hold for Kähler mflds.

Key advantage of Kähler mflds: a theory of harmonic (p,q)–forms. (3.2.6)

Consequences for X compact Kähler:

- Hodge decomposition of cohomology (3.2.12)

$$H^k(X,\mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X) \qquad \overline{H^{p,q}} = H^{q,p}$$

- Serre duality (3.2.12, Exc 3.2.2, generalizes Poincaré duality)

$$H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$$

- Hard Lefschetz (3.3.13)

$$[\omega]^k \cap - : H^{n-k}(X,\mathbb{R}) \xrightarrow{\cong} H^{n+k}(X,\mathbb{R}) \qquad \forall k=0,1,\dots,n$$

§ Line bundles & positivity

Let L line bundle on X.

A hermitian metric h on trivial l.b. $L = X \times \mathbb{C}$

$\Longleftrightarrow$ specify norm of a section, $s(x) = (x, 1)$

$$h_p(\lambda, \mu) = \lambda\bar{\mu}\, h_p(1,1) = \lambda\bar{\mu}\, \|1\|$$

So $h \longleftrightarrow$ a real, positive, smooth fcn $h : X \longrightarrow \mathbb{R}_+$

<u>Def</u>: The (Chern) curvature form of (L, h) is:

$$F = \bar{\partial}\partial \log(h)$$

$$\omega = \frac{i}{2\pi} F \text{ is a closed, real, } (1,1)\text{-form}$$

but <u>not</u> always positive.

Agrees with first Chern class <u>$c_1(L) \in H^2(X, \mathbb{Z})$</u>

(defined via exponential seq.)

$$[\omega] = c_1(L) \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

Def: (L, h) is positive if $\omega = \frac{i}{2\pi} F$ is positive.

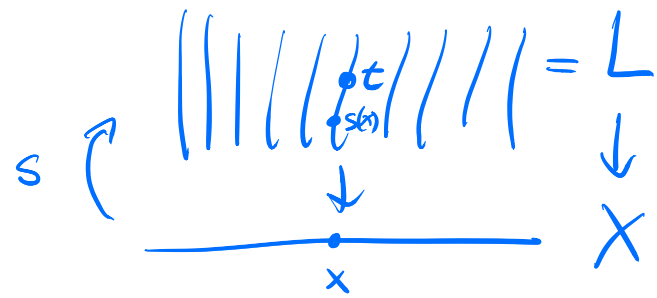
$\iff$ ω is Kähler

Def: L is positive if $\exists h$ s.th. (L,h) positive.

Ex: (4.1.2(i)) $L = \mathcal{O}(1)$ on $\mathbb{P}^n$ is positive:

The coordinate "functions" on $\mathbb{P}^n$ are really sections $s_0 = z_0, \dots, s_n = z_n$ of $\mathcal{O}(1)$. We define the following norm h_p on L_p using some trivialization $\psi: L_p \xrightarrow{\sim} \mathbb{C}$

$$h_p(t) = \frac{|\psi(t)|^2}{\sum_{i=0}^{n} |\psi(s_i)|^2}$$



Well-defined b/c ψ unique up to mult w/ $\lambda \in \mathbb{C}^\times$ and at least one $s_i \neq 0$. (same construction works for any L w/ $s_0, \dots, s_n$ s.th. $\forall p \in X$, $s_i(p) \neq 0$ for some i)

Gives Hermitian metric on L.

Recovers Fubini-Study metric on $\mathbb{P}^n$!

$$\omega_{FS} = \omega(L, h) = \frac{i}{2\pi} \bar{\partial} \partial \log(h)$$

§ Kodaira vanishing and applications

Thm (Kodaira vanishing 5.2.2) X compact complex mfld
If L is positive then:

$$H^q(X, K_X \otimes L) = 0 \quad \forall q > 0,$$

As a corollary (compare proj case) one gets

Thm (Lefschetz hyperplane thm, 5.2.6) If X compact cplx
$D \subset X$ smooth divisor such that $\mathcal{O}(D)$ is positive, then

$$H^k(X, \mathbb{C}) \longrightarrow H^k(D, \mathbb{C})$$

is
- injective $\forall k \leq n-1$
- bijective $\forall k \leq n-2$

Thm (Kodaira embedding thm, 5.3.1) If X compact cplx
and L line bundle, then L positive $\iff$ L ample

difficult part is $\Rightarrow$.

Def: L ample if $\exists N \gg 0$ s.th.

$$X \xrightarrow{|L^N|} \mathbb{P}^M \quad \text{is an embedding}$$

Rmk (implication $\Leftarrow$) If $|L^N|$ embedding, then
$L^N = \mathcal{O}_{\mathbb{P}^M}(1)|_X$ positive (example above) $\Rightarrow$ L positive.

§ Serre vanishing (variant of Kodaira vanishing)

__Thm__ (Serre vanishing, 5.2.7) If X compact cplx mfld and L positive, then $\forall$ vector bundle E on X

$$H^q(X, E \otimes L^N) = 0 \quad \forall q > 0, \quad N \gg 0$$