ex: M differentiable manifolds, $C_M$ = sheaf of diff. func.

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$$C_M(U) = \{ U \xrightarrow{f} \mathbb{R} \ C^\infty \}$$ (a ring)

ex: X complex manifold, $\mathcal{O}_X$ = sheaf of holo fun.

$$\mathcal{O}_X(U) = \{ U \xrightarrow{f} \mathbb{C} \ \text{holo} \}$$ (a ring)

<u>Def</u>: X top. space. A presheaf on X is a contravariant functor

$$\text{Open}(X) \xrightarrow{\mathcal{F}} \text{Sets}$$

that is:

(a) $\forall U \subseteq X$ open, a set $\mathcal{F}(U)$

(b) $\forall V \subseteq U \subseteq X$ open, restriction map $\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ s.t
$s \longmapsto s|_V$

(i) $\rho^U_U = id_{\mathcal{F}(U)}$

(ii) $\forall W \subset V \subset U \subset X$ open $\Rightarrow \rho^U_W = \rho^V_W \circ \rho^U_V$

<u>Def</u>: A presheaf is a sheaf if

(si) if $U = \bigcup_i U_i$, $s, t \in \mathcal{F}(U)$ s.t $s|_{U_i} = t|_{U_i}$ $\forall i \Rightarrow s = t$

(sii) if $U = \bigcup_i U_i$, $s_i \in \mathcal{F}(U_i)$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \Rightarrow$
$\exists s \in \mathcal{F}(U)$ s.t $s|_{U_i} = s_i$

Note: (si) locally equal ⇒ equal
(sii) can give local sections to describe a global section.

Rem:
$$\begin{array}{ccccc} \mathcal{F}(U) & \xrightarrow{\alpha} & \prod_i \mathcal{F}(U_i) & \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} & \prod_{ij} \mathcal{F}(U_i \cap U_j) \\ s & \longmapsto & (s|_{U_i})_i & & \\ & & (s_i)_i & \longmapsto & s_i|_{U_i\cap U_j} \\ & & & \longmapsto & s_j|_{U_i \cap U_j} \end{array}$$

(si) ⟺ $\alpha$ inj
(si)+(sii) ⟺ $\alpha = eq(\beta_1, \beta_2)$
$\alpha$ is the equalizer of $\beta_1$ and $\beta_2$

Def (alternative def of manifold):
- A ringed space is a pair $(X, \mathcal{O}_X)$ of a top. space $X$ and a sheaf of rings $\mathcal{O}_X$ on $X$

get rid of the atlas

"Def": A complex manifold is a ringed space $(X, \mathcal{O}_X)$ s.t
- $\exists X = \bigcup_i U_i$ open covering s.t $\forall i$ $(U_i, \mathcal{O}_X|_{U_i}) \cong (V_i, \mathcal{O}_{V_i})$ where $V_i \subseteq \mathbb{C}^m$ open and $\mathcal{O}_{V_i}$ its sheaf of holo fun.

we haven't defined what's a isom. of ringed spaces ("Def)

Def: A morphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a cont. map $f: X \to Y$ and $f^{\#}: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ that is:
for every $V \subseteq Y$ open, a map
$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$$
$$g \longmapsto g \circ f$$

observe that
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$
$g$ holo ⇒ $gf$ holo

Th: These two def of manifolds agree (both for diff and holo).

- Def: Let $\mathcal{F}$ be a sheaf. A stalk of $\mathcal{F}$ is
$$\mathcal{F}_x := \{(U, s) \mid x \in U \text{ nbhd}, s \in \mathcal{F}(U)\}/\sim$$
$(U,s)$: a germ

$(U_1, s_1) \sim (U_2, s_2)$ if $\exists U_3 \ni x$ nbhd s.t $U_3 \subseteq U_1 \cap U_2$ and $s_1|_{U_3} = s_2|_{U_3}$

We have a map
$$\mathcal{F}(U) \longrightarrow \mathcal{F}_x$$
$$s \longmapsto (U, s)$$

Observe that if $\mathcal{F} = \mathcal{O}_X$, we denote $\mathcal{O}_{X,x} := (\mathcal{O}_X)_x$ the stalk

ex: $\mathcal{O}_{\mathbb{C}^m, 0} = \{f : U \longrightarrow \mathbb{C}$ holo for some nbhd $U$ of $0\} =$
$= \{$power series $\sum a_\alpha z^\alpha$ converges in a nbhd of $0\}$

where $z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ with $\alpha \in \mathbb{N}^m$, $a_\alpha \in \mathbb{C}$.

and $a_\alpha = \frac{(\partial_\alpha f)(0)}{\alpha!}$, $\partial_\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_m}\right)^{\alpha_m}$, $\alpha! = \alpha_1! \cdots \alpha_m!$

Furthermore, observe that if $X$ is a complex manifold,

$\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^m,0}$ because the open sets of $X$ are homeomorphic to open sets of $\mathbb{C}^m$.

__Rem__: $f, g$ holo func on $U \subseteq X$, then

$$f_x = g_x \underset{\not\Leftarrow}{\Rightarrow} f(x) = g(x)$$
$$\Leftrightarrow (\partial_\alpha f)(x) = (\partial_\alpha g)(x) \ \forall \alpha \in \mathbb{N}^m$$

EX: $\mathcal{O}_{X,x}$ is an integral domain ($fg = 0 \Rightarrow f = 0$ or $g = 0$)

↳ and also a UFD and noetherian (use Weierstrass preparation th)
(more challenging)

__Def__: $f: X \to Y$ is biholomorphic if it is holomorphic and has a holomorphic inverse.

__Prop__ (1.1.13): $f$ holomorphic and bijective $\Leftrightarrow$ $f$ biholo.

ex: The analogue is not true for diff. functions:

$f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$  $f'(0) = 0$ · so $(f^{-1})'(0) \not\exists$ so $f^{-1}$ not smooth (because of the jacobian condition)

__Prop__ (2.1.5): If $X$ connected cpt complex manifold, then $\mathcal{O}_X(X) =: \Gamma(X, \mathcal{O}_X) = \mathbb{C}$

that is, every global holo function is constant.

__Pf__ $f: X \to \mathbb{C}$ holo

$X$ cpt $\Rightarrow$ $f(X)$ cpt. Thus $|f|: X \to \mathbb{R}$ attains a max $M$.

impossible unless $f$ is constant. (see what follows)

__Th__ (maximum modulus principle)

$X$ connected complex manifold, $f: X \to \mathbb{C}$ holo, not constant, then $|f|$ has no (local) maximum.

__Pf__ Let $U \subseteq X$ be a chart

$$\begin{array}{c} U \subseteq X \\ \cong \downarrow \\ V \subseteq \mathbb{C}^m \end{array}$$

If $f$ has a maximum $\underset{\text{HHP for } V}{\Rightarrow}$ $f|_V$ constant $\underset{\text{identity th.}}{\Rightarrow}$ $f$ constant

__Th__ (HHP for $\mathbb{C}^m$): Let $U \subseteq \mathbb{C}^m$ connected open, $f: U \to \mathbb{C}$ holo non-constant. Then $f$ has no maximum. If $U$ bounded and $f$ extends continuously to $\overline{U}$ then $f$ attains a maximum on $\partial U$.

Pf Suppose $z_0 \in U$ maximum $f(z_0)=M$. Pick $B_\varepsilon(z_0) \subseteq U$.
Then $|f(z)| \leq |f(z_0)|$ on $\partial B_\varepsilon(z_0)$, hence Cauchy's int. formula:

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z-z_0}\,dz$$

$$\Rightarrow |f(z_0)| \leq \frac{1}{2\pi}\int_{\partial B_\varepsilon(z_0)} \frac{M}{|\varepsilon|}\,dz = M$$

$$\Rightarrow f|_{B_\varepsilon(z_0)} = M \underset{\text{identity th}}{\Longrightarrow} f(z) = M$$

For the last claim: $|f| : \underset{\text{cpt}}{\overline{U}} \longrightarrow \mathbb{R}$ has maximum.

- Th (identity th): $X$ connected complex manifold
$f, g : X \longrightarrow \mathbb{C}$ holo. $\exists W \subseteq X$ open s.t. $f|_W = g|_W \Rightarrow f = g$

Note: If exists a stronger version.

- Pf Let $S = \{z \in X \mid f_z = g_z\}$. stalks

(i) $S$ is open (easy)
(ii) $S$ is closed (it has to be checked in charts. Can assume $X \subseteq \mathbb{C}^n$)

$$S = \{z \in X \mid (\partial_\alpha f)(z) = (\partial_\alpha g)(z)\ \forall \alpha\} = \bigcap_\alpha \{z \in X \mid \partial_\alpha f(z) = \partial_\alpha g(z)\} =$$

$$= \bigcap \text{closed} = \text{closed}$$

(iii) $S \neq \emptyset$ because $f_x = g_x \ \forall x \in W$

$\underset{X \text{ connected}}{\Longrightarrow} S = X$

- Meromorphic functions

Def 1: $X$ complex manifold. A meromorphic function is
(a) $S \subseteq X$ nowhere dense set ($\overline{S}^{\,int} = \emptyset$)
(b) $f : X \setminus S \longrightarrow \mathbb{C}$ holo

s.t. $\exists X = \bigcup_i U_i$ open covers, and holo $g_i, h_i : U_i \longrightarrow \mathbb{C}$ s.t.

$$f|_{U_i \setminus S} = (g_i/h_i)|_{U_i \setminus S}$$

Rem: Can take $S \cap U_i = Z(h_i)$.
$f$ extends uniquely to $U_i \setminus Z(h_i)$ (identity th)

Th (Oka): If $X \subseteq \mathbb{C}^n$ open, then every meromorphic fun. can be represented by $g, h : X \longrightarrow \mathbb{C}$ holo as $f = g/h$ and $S = \cancel{\ }Z(h)$.

Note: The proof is not trivial but very analytic ex: $m=1$ $\quad f(z)=\dfrac{g(z)}{h(z)}$ $\quad S=Z(h)$ discrete

locally: $f(z)=(z-z_0)^{-d}\,k(z)$ with $k$ holomorphic

at $z_0\in S$

ex: $f(z)=\dfrac{1}{\sin z}$ $\quad S=2\pi\mathbb{Z}$

ex: $f(z)=e^{1/z}$ holo outside origin but not meromorphic

ex: $\zeta(z)=\underbrace{\sum_{m\geq 0} m^{-s}}_{\text{conver. for } Re(s)>1}$ has meromorphic cont. to $\mathbb{C}$ $\quad S=\{1\}$

Riemann zeta-fun

<u>Fact</u> (next time):

meromorphic fun on $U\subset\mathbb{C}^1$ $\longleftrightarrow$ holomorphic fun $U\longrightarrow\mathbb{P}^1_{\mathbb{C}}$

$f=\dfrac{g}{h}\longmapsto\big(z\longmapsto(g(z):h(z))\big)$

chosen st. $Z(g)\cap Z(h)=\emptyset$

(because otherwise we can factor the common part out)

where $\mathbb{P}^1_{\mathbb{C}}=\mathbb{C}^2\setminus\{0\}/\mathbb{C}^\times$

ex: $m=2$ $\quad f(z)=\dfrac{z_1}{z_2}$