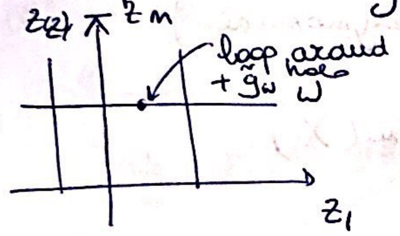
11/02/20

Prop (Riemann existence th): and $Z(f)$ nowhere dense (3a)
$U \subseteq \mathbb{C}^m$, $f: U \longrightarrow \mathbb{C}$ holo, $g: U \setminus Z(f) \longrightarrow \mathbb{C}$ holo + loc. bounded at $Z(f)$ $\Longrightarrow \exists!\ \tilde{g}: U \longrightarrow \mathbb{C}$ holo extension of g.

Cor: X cplex manifold, Z analytic subspace of X, (locally cut out by holo functions), $g: X \setminus Z \longrightarrow \mathbb{C}$ holo + locally bounded
$\Longrightarrow \exists!\ \tilde{g}: X \longrightarrow \mathbb{C}$ holo extension of g.

Pf (Prop)
Reduction to the one variable case.
$g_\omega(z_1) := g(z_1, \omega)$ with $\omega = (z_2, z_3, \ldots z_m)$
RET in 1 variable $\Longrightarrow g_\omega$ extends to $\tilde{g}_\omega : U \cap \{(z_2, \ldots z_m) = \omega\} \longrightarrow \mathbb{C}$
Then prove that $\tilde{g}(z_1, \omega) := \tilde{g}_\omega(z_1, \omega)$ holo
We do it by using Cauchy's integral formula



Cor: X connected, $Z \subseteq X$ analytic subspace $\Longrightarrow X \setminus Z$ connected.

Pf Suppose $X \setminus Z = U_1 \amalg U_2$ disconnected
$g: X \setminus Z \longrightarrow \mathbb{C}$ s.t $g|_{U_1} = 1$ and $g|_{U_2} = 0$
$\overset{RET}{\Longrightarrow} \tilde{g}: X \longrightarrow \mathbb{C}$ holo and $\tilde{g}|_{U_2} = 0 \underset{\text{id. th}}{\Longrightarrow} \tilde{g} = 0$ $\lightning$

For (3b) $K(X_1 \amalg X_2) = K(X_1) \times K(X_2)$ product of fields (not a field)

(3c) $K(X) \xrightarrow[\text{if } X \text{ conn.}]{1\text{-}1} \mathrm{Frac}(\mathcal{O}_{X,x})$ ring homo
$f \longmapsto g_x / h_x$ if $f = g/h$ near x
well def. if $h_x \neq 0$ (S nowhere dense)

Def: $\mathcal{K}_X$ sheaf of meromorphic functions
$\mathcal{K}_X(U) := K(U)$ meromorphic on U
$(\mathcal{K}_X)_x = \mathcal{K}_{X,x} = \mathrm{Frac}(\mathcal{O}_{X,x})$
ex: $f(z) = e^{1/z}$, $f_1 \in \mathcal{O}_{\mathbb{C},1}$ but it does not come from $K(\mathbb{C})$

Def#2: A meromorphic function is a function

$$f: X \longrightarrow \coprod_{x\in X} \mathrm{Frac}(\mathcal{O}_{X,x})$$
$$x \longmapsto \left(\cancel{g_x}/\cancel{h_x}\right)_{x\in X}$$

s.t $\forall x\in X$ $\exists U \ni x$ nbhd and holo g, h s.t $g_y/h_y = f(y)$ $\forall y \in U$

Note: It means that a meromorphic function is determined by its germs

Def: X cpt connected. The algebraic dimension of X is

$$a(X) = \mathrm{trdeg}_{\mathbb{C}} K(X)$$ thought as the constant fun: $\mathbb{C} \subseteq K(X)$

Prop (Siegel): $0 \le a(X) \le \dim(X)$

Fact: X algebraic (ex: locally zero-locus of polynomials)

$\Rightarrow a(X) = \dim(X)$

Def: X cpt conn, Moishezon if $a(X) = \dim(X)$

Th (Artin): X algebraic $\Leftrightarrow a(X) = \dim X$

Note: This is characterization of algebraic sets as those sets having "many" meromorphic functions.

- Projective space

$$\mathbb{P}^m = \mathbb{P}^m_{\mathbb{C}} = \mathbb{C}^{m+1} \setminus \{0\} / \mathbb{C}^\times$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is a group under multiplication.

$\mathbb{C}^\times$ acts on $\mathbb{C}^{m+1}$; $\lambda z = (\lambda z_0, \ldots \lambda z_m)$

$\lambda \in \mathbb{C}^\times$, $z = (z_0, \ldots z_m) \in \mathbb{C}^{m+1}$

orbit of z ~~are~~ is denoted as $(z_0 : \ldots : z_m)$

A slightly different perspective is given by:

$$Gr_1(\mathbb{C}^{m+1}) = \{L \subseteq \mathbb{C}^{m+1} \mid \text{linear 1-dim subspaces}\} \cong \mathbb{P}^m$$

$L \longleftrightarrow L \setminus \{0\}$

ex: $\mathbb{P}^1$ = Riemann sphere $= \mathbb{C} \cup \{\infty\}$

$\{(1 : z_1)\}$ ↗ $\mathbb{C}$, ∞ = $(0:1)$

$\infty = (0:1)$

$\simeq S^2$

$0 = (1:0)$

We now study the structure of cplex manifold

Topology: $\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n \supseteq U$

U is open $\Leftrightarrow$ $\pi^{-1}(U)$ is open (quotient topology)

Standard open covering: $U_i = \{z_i \neq 0\} \subseteq \mathbb{P}^n$

$\pi_i^{-1}(U_i) = \{z_i \neq 0\} \subseteq \mathbb{C}^{n+1} \setminus 0$

$\Rightarrow \mathbb{P}^n = \bigcup_{i=0}^{n} U_i$

$\varphi_i : U_i \longrightarrow \mathbb{C}^n$

$(z_0 : \_ : z_n) \longmapsto \left(\frac{z_0}{z_i}, \widehat{\frac{z_i}{z_i}}, \frac{z_n}{z_i}\right)$

$\left(\frac{z_0}{z_i} : \_ : \frac{z_n}{z_i}\right) = (z_0 : \_ : z_n)$, $\quad \frac{z_0}{z_i} = x_0,\ \hat{x}_i,\ \frac{z_n}{z_i} = x_n$

$\frac{z_i}{z_i} = 1$

$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$

$\varphi_i(U_i \cap U_j) = \{(x_0, \_ \hat{x}_i, \_ x_n) \mid x_j \neq 0\} \subseteq \mathbb{C}^n$

$\varphi_j(U_i \cap U_j) = \{(x_0, \_ \hat{x}_j, \_ x_n) \mid x_i \neq 0\} \subseteq \mathbb{C}^n$

$(x_0, \_ \hat{x}_i, \_ x_n) \longmapsto \cancel{\left(\frac{x_0}{x_j}, \_ \overset{j\text{-th}}{1}, \_ \frac{x_n}{x_j}\right)}$

$\left(\frac{x_0}{x_j}, \_ \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \_ \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \_ \frac{x_n}{x_j}\right)$

ex: $n = 1$

$U_0 = \{(x_0 : x_1) \mid x_0 \neq 0\} \cong \mathbb{C}$

$(x_0 : x_1) \longmapsto \frac{x_1}{x_0}$

$U_1 = \{(x_0 : x_1) \mid x_1 \neq 0\} \cong \mathbb{C}$

$(x_0 : x_1) \longmapsto \frac{x_0}{x_1}$

$U_0 \cap U_1 \cong \mathbb{C}^\times \xrightarrow{\varphi_1 \circ \varphi_0^{-1}} \mathbb{C}^\times$

$z \longmapsto \frac{1}{z}$

Rem: • $\mathbb{P}^n$ Hausdorff

• $\mathbb{P}^n$ cpt and connected

$S^{2n+1} \subseteq \mathbb{R}^{2n+2} \setminus 0 = \mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n \Rightarrow S^{2n+1} \cong \mathbb{P}^n$

cpt conn.

su

Algebraic functions:

homogeneous polynomials of deg d:

$$f(z) = \sum_{|\alpha|=d} a_\alpha z^\alpha \quad \text{s.t } f(\lambda z) = \lambda^d f(z)$$

$$Z(f) = \{(z_0 : \_ : z_m) \mid f(z) = 0\}$$

rational hom. fun. of deg 0:

$$f(z) = \frac{p(t)}{q(t)} \qquad p, q \text{ hom. pol. of } \underline{\deg d}$$

we need it because we need that λ cancelled it out

$$\Rightarrow f(\lambda z) = f(z)$$

gives merom. fun.

How many rational functions?

$$\frac{z_1}{z_0}, \frac{z_2}{z_0}, \_ \frac{z_m}{z_0} \quad \text{alg. indep.}$$

$$f(z) = \frac{p(z)}{q(z)} = \frac{\cancel{z_0^d}\, p(1, \frac{z_1}{z_0}, \_ \frac{z_m}{z_0})}{\cancel{z_0^d}\, q(1, \frac{z_1}{z_0}, \_ \frac{z_m}{z_0})}$$ ← it is a rational fu in z_i

by def $\deg p = \deg q = d$

Recall:

Chow's th: Every $\underbrace{\text{analytic subset}}_{\text{locally } Z(\text{holo})}$ of $\mathbb{P}^m$ is $\underbrace{\text{algebraic}}_{\text{loc } Z(\text{poly})}$

Cor: Every meromorphic fun. of $\mathbb{P}^m$ is rational.

"Pf" Let $f: \mathbb{P}^m \setminus S \longrightarrow \mathbb{C}$ meromorphic

$S = P(f)$ = set of poles, locally $Z(h_i)$ if $f = g_i/h_i$ ⇒ analytic subspace

$Z(f)$ = set of zeros " $Z(g_i)$ " "

$$\overset{\text{Chow}}{\rightsquigarrow} \quad Z(f) = Z(p) \quad p \text{ hom. poly}$$

$$P(f) = Z(q) \quad q \text{ hom. poly}$$

⇒ $f\frac{q}{p}$ no zeros nor poles ⇒ constant

⇒ holo ⇗

⇒ $f = \frac{p}{q} \cdot$ constant is rational.

Problem here: is $\deg p = \deg q$? So that p/q is meromorphic.

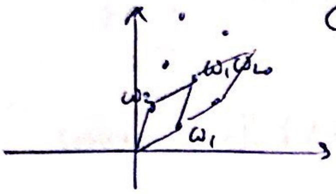
- Complex tori:

$X = \mathbb{C}^g / \mathbb{Z}^{2g}$ ← makes sense because $\Lambda := \mathbb{Z}^{2g} \subseteq \mathbb{C}^g = \mathbb{R}^{2g}$ ← lattice

$\mathbb{Z}^{2g} = \mathbb{Z}\langle \omega_1, \_ \omega_{2g} \rangle = \omega_1\mathbb{Z} + \_ + \omega_{2g}\mathbb{Z}$

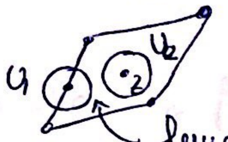
in such a way that $\mathrm{span}\{\omega_1, \_ \omega_{2g}\} = \mathbb{R}^{2g}$

ex: $g = 1$ $\quad \mathbb{C} = \mathbb{R}^2$



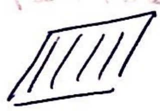
$\omega \in \Lambda$ acts on $\mathbb{C}^g$ simply by addition: $\omega . z = \omega + z$

Manifold: $U_i \overset{\varphi_i}{\subset} \mathbb{C}^g$ are the charts



$\Rightarrow$ Hausdorff, dim $= g$

cpt and conn:



$\longrightarrow\!\!\!\!\to \mathbb{C}^g / \Lambda$

$\shortparallel$

$[0,1]^{2g}$

Note: In algebraic geom it is an abelian variety

Addition of $\mathbb{C}^g \longrightarrow$ addition on $\mathbb{C}^g/\Lambda$

Note: In diff. geom/top: $\mathbb{C}^g/\mathbb{Z}^{2g} \cong (\mathbb{R}/\mathbb{Z})^{2g} \cong$
$\cong (S^1)^{2g}$
$\Rightarrow$ g-dim. torus

ex: $g = 1$ (elliptic curves in alg. geom.)

$\mathbb{C}/\Lambda$ for $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$

$f(\Lambda_1) = \Lambda_2$
$f : \mathbb{C}^n \to \mathbb{C}^n$ lin, bij
$\mathbb{C}^n/\Lambda_1 \xrightarrow{\cong} \mathbb{C}^n/\Lambda_2$

Not all ω_1 and ω_2 gives different $\mathbb{C}/\Lambda$

~~ex~~ for example $\frac{1}{\omega_1}$ on $\mathbb{C}$ gives:

$$\mathbb{C}/\omega_1\mathbb{Z} + \omega_2\mathbb{Z} \cong \mathbb{C}/\mathbb{Z} + \frac{\omega_2}{\omega_1}\mathbb{Z}$$

but $(\omega_1, -\omega_2)$ gives same cplx torus

$$\omega_1\mathbb{Z} + \omega_2\mathbb{Z} = (c\omega_1 + d\omega_2)\mathbb{Z} + (a\omega_1 + b\omega_2)\mathbb{Z} \quad \text{s.t}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2)$$

Def: Weierstrass function $\quad p(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$

↑ meromorphic func. having poles in every lattice point

Note: The convergence is guaranteed by the addition of $-1/\omega^2$.

Note:

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d, dashed, "p"'] & \mathbb{C}/\Lambda \arrow[d, dashed, "\text{merom.}"] \\ \mathbb{C} & \mathbb{C} \end{tikzcd}$$

derivative

$$f: \mathbb{C}/\Lambda \longrightarrow \mathbb{P}^2$$

$$z+\Lambda \longmapsto \begin{cases} (p(z) : p'(z) : 1) & \text{if } z \in \Lambda \\ (0:1:0) & \text{if } z \in \Lambda \end{cases}$$

If $g = 1$: every cplx torus is algebraic

If $g > 1$: $0 \leq a(\mathbb{C}^g/\Lambda) \leq g$.

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