If $g > 1$: $0 \leq a(\mathbb{C}^g/\Lambda) \leq g$.

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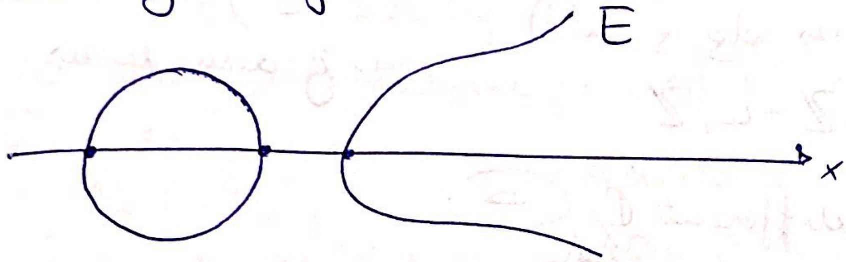
Comments on HW#3:

$$\mathbb{C}/\Lambda \xrightarrow{g} \mathbb{P}^2$$
$$z \longmapsto \begin{cases} (\wp(z):\wp'(z):1) & \text{if } z \notin \Lambda \\ (0:1:0) & \text{if } z \in \Lambda \end{cases}$$

$$\mathrm{Im}(g) \subseteq E = Z(y^2 z - (4x^3 - g_2 x z^2 - g_3 z^3))$$

Fact: g is bijective onto E



[something missing]

Note: $\sum_{w \in \Lambda \setminus \{0\}}$ the order in the sum is not specified because the sum is absolutely convergent (so the order doesn't count).

- Submanifold

"Def": A submanifold $Y \subseteq X$ locally looks like $\mathbb{C}^k \subseteq \mathbb{C}^n$
$(z_1, \dots z_k) \longmapsto (z_1, \dots z_k, 0, \dots$

Def: X cplex manifold of dim n. $Y \subseteq X$ subset is a submanifold of dimension k if $\forall y \in Y$ $\exists$ $y \in U \subseteq X$ open nbhd and a holo chart $\varphi: U \longrightarrow \mathbb{C}^n$ s.t $Y \cap U = \varphi^{-1}(\mathbb{C}^k)$

Def: Y has codimension $n-k$

Def: A cplex manifold X is projective if it's biholomorphic to a submanifold of $\mathbb{P}^N$ for some N.

Rem: $(Y \cap U, \varphi|_{Y \cap U})$ chart of Y at y.

ex: $\mathbb{C}/_\Lambda \cong E \subset \mathbb{P}^N$ projective submanifold.

Why is E a submanifold?

We have to look at the jacobian:

$$f: \underset{\underset{\mathbb{C}^m}{\cap}}{U} \longrightarrow \mathbb{C}^n \qquad J(f)(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)$$

where $J(f): T_z U \longrightarrow T_{f(z)}\mathbb{C}^n$ (as in diff. geom.)

$$T_zU = \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}\right\rangle = \mathbb{C}^m \qquad T_{f(z)}\mathbb{C}^n = \left\langle \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n}\right\rangle = \mathbb{C}^n$$

- Real jacobian of f: $z_j = x_j + i y_j$, $w_i = u_i + i v_i$

$$J_{\mathbb{R}}(f)(z) := \begin{pmatrix} \frac{\partial u_i}{\partial x_j} & \frac{\partial u_i}{\partial y_j} \\ \frac{\partial v_i}{\partial x_j} & \frac{\partial v_i}{\partial y_j} \end{pmatrix}$$

real $(2n \times 2m)$-matrix basis $\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\rangle$ and $\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i}\right\rangle$

- with complex basis $\left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\right\rangle$ and $\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar w}\right\rangle$

$$J_{\mathbb{R}}(f)_{\mathbb{C}}(z) = \begin{pmatrix} \frac{\partial f_i}{\partial z_j} & \frac{\partial f_i}{\partial \bar z_j} \\ \frac{\partial \bar f_i}{\partial z_j} & \frac{\partial \bar f_i}{\partial \bar z_j} \end{pmatrix} = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$$

complex $(2n \times 2m)$-matrix

- if $m=n$ $\Rightarrow$ $\det J_{\mathbb{R}}(f) = |\det(J(f))|^2 \geq 0$ (cplx manifold are always orientable)

  if $m \neq n$ $\Rightarrow$ $\mathrm{rk}(J_{\mathbb{R}}(f)) = 2\,\mathrm{rk}(J(f))$

- <u>Def</u>: $z \in U$ is a regular point if $\mathrm{rk}(J(f)(z)) = n$

  $w \in \mathbb{C}^n$ is a regular value if z regular point $\forall z \in f^{-1}(w)$.

Implicit function th for holo functions:

$$f: \underset{\underset{\mathbb{C}^m}{\cap}}{U} \longrightarrow \mathbb{C}^n \quad \text{holo}$$

(i) if $m \geq n$, regular at $z_0 \in U$, then f locally looks like (around z_0)

$$p: \mathbb{C}^m \longrightarrow \mathbb{C}^n$$
$$(z_1, \dots z_m) \longmapsto (z_1, \dots z_n) \quad \text{for } z_0 = 0$$

To be precise: $\exists \underset{\underset{z_0}{\Psi}}{U'} \subseteq U$, $\underset{\underset{\mathbb{C}^m\ 0 \longmapsto z_0}{\cap}}{V} \xrightarrow[\text{biholo}]{g} U'$ s.t $f \circ g = p$

(ii) if $m \leq n$, $rk(J(f)(z_0)) = m$ (jacobian as big as possible)

then locally looks like $i: \mathbb{C}^m \hookrightarrow \mathbb{C}^n$

$$(z_1, \dots z_m) \longmapsto (z_1, \dots z_m, 0, \dots 0)$$

In case (i): $f^{-1}(0)$ locally looks like $\{(0, \dots 0, z_{M+1}, \dots z_m)\}$

so a submanifold locally at z_0. ┘

Hypersurfaces are examples of submanifolds:

$f: X \longrightarrow \mathbb{C}$ holo

If 0 is a regular value of f, then $Z(f) \subseteq X$ is a closed submanifold

In general, $Z(f)$ is an analytic subspace,

$Z(f)_{reg} \subseteq Z(f)$ subset of points where f regular

$Z(f)_{sing} := Z(f) \setminus Z(f)_{reg}$

so $z \in Z(f)$ singular exactly when $rk(J(f)(z)) < 1$

$\iff f(z) = 0, \ \frac{\partial f}{\partial z_i}(z) = 0 \quad \forall i$

So $Z(f)_{sing} = Z(f, \frac{\partial f}{\partial z_1}, \dots \frac{\partial f}{\partial z_m})$ analytic subspace.

ex: $f = y^2 - x^3 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ cusp singularity

$Z(f) \subseteq \mathbb{C}^2$

$\frac{\partial f}{\partial x} = -3x^2 \qquad \frac{\partial f}{\partial y} = 2y$

$Z(f)_{sing} = \{(0,0)\}$ both have to vanish simultaneously

ex: $f = y^2 - x^3 + 1$

$Z(f)_{sing} = \{x = 0, y = 0, y^2 - x^3 + 1 = 0\} = \emptyset$

If f is a homogeneous polynomial (of deg d), $Z(f) \subseteq \mathbb{P}^n$ makes sense.

$$Z(f) \cap U_i \subseteq U_i = \{(z_0 : \dots : z_m) : z_i \neq 0\}$$

$$\downarrow \qquad \varphi_i \downarrow \cong \qquad \downarrow$$

$$\varphi_i(Z(f) \cap U_i) \subseteq \mathbb{C}^n \quad \left(\frac{z_0}{z_i}, \dots \frac{z_M}{z_i}\right)$$

where projective hypersurfaces ~~sett~~ come from.

$\parallel$

$Z(\tilde{f}(x_0, \dots x_m))$ where $f(z_0, \dots z_m) = z_i^d \tilde{f}\left(\frac{z_0}{z_i}, \dots \frac{z_m}{z_i}\right)$

EX: $\tilde{f}\left(\frac{z_0}{z_i}, \dots \frac{z_m}{z_i}\right)$ is regular at $z = (z_0 : \dots : z_m) \in U_i$

$\iff f(z_0, \dots z_m)$ is regular at $z = (z_0, \dots z_m) \in \mathbb{C}^{M+1} \setminus \{0\}$

$$\begin{tikzcd} \mathbb{C}^{m+1}\setminus\{0\} \arrow[r, "f"] \arrow[d] & \mathbb{C} \\ U_i \subseteq \mathbb{C}^{m+1}\setminus\{0\}/\mathbb{C}^* = \mathbb{P}^m \arrow[ur, bend right, "\tilde{f}"'] & \end{tikzcd}$$

$f_1, \dots f_k : U \longrightarrow \mathbb{C}$ can be seen as

$f : U \longrightarrow \mathbb{C}^k$

$Z = Z(f) = f^{-1}(0) = Z(f_1) \cap \dots \cap Z(f_k)$

0 regular value of f, then Z is a submanifold of codim k (complete intersections).

ex: $f = y^2 - (4x^3 - g_2 x - g_3)$ elliptic curve

$\frac{\partial f}{\partial x} = -12x^2 + g_2 \qquad \frac{\partial f}{\partial y} = 2y$

$Z(f)_{sing} = \{y = 0,\ g_2 = 12x^2,\ g_3 = 4x^3 - g_2 x = 8x^3, \} \Delta = 0 \}$

$\Delta = g_2^3 - 3^3 g_3^2 \neq 0$ ← discriminant ← but

$\hat{\ }$ EX of Riemann-Zeta function

- Structure sheaves

$Y \overset{i}{\subseteq} X$ submanifold (closed)

$\mathcal{O}_X$ sheaf of holo functions

$\mathcal{O}_X(U) = \{ f : U \longrightarrow \mathbb{C} \text{ holo} \} \quad \forall U \subseteq X$ open

We have something similar for Y:

$i_* \mathcal{O}_Y$ = push-forward sheaf on X

$(i_*\mathcal{O}_Y)(U) := \mathcal{O}_Y(i^{-1}(U)) = \mathcal{O}_Y(U \cap Y) = \{ f : U \cap Y \longrightarrow \mathbb{C} \text{ holo} \}$

$\forall U \subseteq X$ open

There is a homomorphism of sheaves of rings:

$$\mathcal{O}_X \xrightarrow{\alpha} i_* \mathcal{O}_Y$$

$$f \longmapsto f|_Y$$

$$\mathcal{O}_X(U) \xrightarrow{\alpha(U)} (i_*\mathcal{O}_Y)(U)$$

$$(f : U \to \mathbb{C}^m) \longmapsto (f|_{U\cap Y} : U \cap Y \longrightarrow \mathbb{C}^m)$$

Warning: $\alpha(U)$ need not be surjective. holo fun on $U \cap Y$ that cannot be extended to U.

ex: $X = \mathbb{P}^1$, $Y = \{0\} \cup \{1\} = Z(x(x-y))$.
$\qquad\qquad\quad (0:1) \quad (1:1)$

$\mathcal{O}_X(X) = \mathbb{C}$ $\qquad$ f $\qquad$ not surjective

$(i_* \mathcal{O}_Y)(X) = \mathcal{O}_Y(Y) = \mathbb{C} \times \mathbb{C}$ $\quad (f(0), f(1))$

But α is a surjective homo of sheaves!

Def: $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ homo of sheaves is ~~surjective~~

- injective if $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ inj $\forall U \subseteq X$
- surjective if "locally surjective", i.e
  $\forall U \subset X$, $\forall s \in \mathcal{G}(U)$ $\exists V \subset U$, $\exists t \in \mathcal{F}(V)$ s.t $\varphi(t) = s|_V$

Fact: $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is inj $\iff$ $\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x$ inj $\forall x$
$\qquad\qquad\qquad$ surj $\iff$ $\qquad\qquad$ surj $\forall x$

Lemma: $\alpha : \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$ is surjective.

Pf Pick $x \in X$, $\exists$ chart $U \subseteq X$ s.t $Y \cap U = \varphi^{-1}(\mathbb{C}^k)$ for the def of submanifold.
$\qquad\qquad\qquad\qquad \downarrow \varphi$
$\qquad\qquad\qquad\qquad \mathbb{C}^n$

$$\begin{array}{ccc} Y\cap U & \subset & U \\ \downarrow\cong & & \downarrow\cong \\ V\cap\mathbb{C}^k & \subset & V \\ \cap & & \cap \\ \mathbb{C}^k & \subset & \mathbb{C}^n \end{array}$$

Enough to prove that $f : V \cap \mathbb{C}^k \longrightarrow \mathbb{C}$ holo extends to $\tilde{f} : V \longrightarrow \mathbb{C}$ (using $\cong$ on the left)
Let $\tilde{f}(z_1, \ldots z_n) = f(z_1, \ldots z_k)$

This gives a short exact sequence of sheaves on X:

$$0 \longrightarrow \underset{\ker\alpha}{\underset{\|}{\mathcal{I}_Y}} \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} i_* \mathcal{O}_Y \longrightarrow 0$$

i.e, α is surjective as morphism of sheaves;
$(\ker\alpha)(U) = \ker(\alpha(U))$

Concretely,
$\mathcal{I}_Y(U) = \{ f : U \longrightarrow \mathbb{C}$ holo : $f|_{U\cap Y} = 0 \}$ is an ideal sheaf because α is a ring homo. $\blacksquare$

Note: If Y is not a submanifold, $i_* \mathcal{O}_Y$ doesn't make sense, but $\mathcal{I}_Y$ does.
ex: $Y = Z(f)$, then $\mathcal{I}_Y = f\mathcal{O}_X$ makes sense even if $Z(f)$ is not regular.