Next time: constant sheaf, sheaf cohomology (resolutions, soft sheaves, de-Rham-sheaf cohom., ...)

[H, Appendix B] ← have a look at this

25/02/20

- Vector bundles

X cplex manifold of rank r

Def: A vector bundle is a cplx manifold E with a holo map $\pi: E \longrightarrow X$ and $\forall x \in X$, a structure of cplx vector space of dim r on

$$E(x) := \pi^{-1}(x)$$

s.t $\exists X = \bigcup U_i$ open cover and biholo maps

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i", "\cong"'] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C}^r \arrow[dl, "pr_I"] \\ & U_i & \end{tikzcd}$$

s.t the induced $E(x) \xrightarrow[\cong]{(\varphi_i)(x)} \mathbb{C}^r$ is $\mathbb{C}$-linear.

Note: $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ locally looks like $\begin{array}{c} X \times \mathbb{C}^r \\ \downarrow pr_1 \\ X \end{array}$.

Over $U_{ij} = U_i \cap U_j$ we have transition ~~maps~~ functions:

$$\psi_{ij} = \varphi_i \circ \varphi_j^{-1} : U_{ij} \times \mathbb{C}^r \xrightarrow{\cong} U_{ij} \times \mathbb{C}^r$$

and $\psi_{ij}(x): \mathbb{C}^r \xrightarrow{\cong} \mathbb{C}^r$ is $\mathbb{C}$-linear $\Longleftrightarrow \tilde{\psi}_{ij}(x) \in GL_r(\mathbb{C})$

$\Longleftrightarrow$ $\begin{array}{rcl} U_{ij} & \xrightarrow{\tilde{\psi}_{ij}} & GL_r \\ x & \longmapsto & \tilde{\psi}_{ij}(x) \end{array}$ holo $\left(\underset{\{\det \neq 0\}}{GL_r(\mathbb{C})} \overset{\text{open}}{\subseteq} \mathbb{C}^{r^2} \text{ submanifold} \right)$ ← that's why it makes sense to talk about holo fun.

Rem: Over $U_{ijk} := U_i \cap U_j \cap U_k$

$$\psi_{ij} \circ \psi_{jk} \circ \psi_{ki}\big|_{U_{ijk}} = id, \quad \tilde{\psi}_{ij} \tilde{\psi}_{jk} \tilde{\psi}_{ki} = I$$ ← "cocycle condition"

such $\tilde{\psi}_{ij}$ are called cocycles

Later we will see that $\{\tilde{\psi}_{ij}\}/_{\sim} = \check{H}^1(\{U_i\}, GL_r)$ ← Čech cohomology

ex: (trivial vector bundle)

$\begin{array}{c} E = X \times \mathbb{C}^r \\ \downarrow pr_1 \\ X \end{array}$ We can consider the trivial covering with $\psi_{ij} = id$, or we can choose non-trivial coverings, where some "twist" appears:

ex: (holomorphic tangent bundle)

$$\begin{array}{cc} T_x X & \tau_X \\ \downarrow & \downarrow \\ x \in & X \end{array}$$ rk n v.b, where $n = \dim X$, and $T_x X = \tau_X(x) = \mathbb{C}^n$

<u>Note</u>: $\{ \text{v.b of rank } r \} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{l} X = \bigcup U_i \text{ open cover} \\ \text{cocycle } \tilde{\psi}_{ij} \end{array} \right\} / \sim$

Recall that: $z \in \mathbb{C}^n$, $T_z \mathbb{C}^n \cong \mathbb{C} \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \rangle \cong \mathbb{C}^n$

$U \subseteq \mathbb{C}^n$; $\tau_U \cong U \times \mathbb{C} \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \rangle \cong U \times \mathbb{C}^n$

So $X = \bigcup U_i$ holo atlas, $U_i \xrightarrow[\cong]{\varphi_i} V_i \subseteq \mathbb{C}^n$

$$\psi_{ij} : \tau_{U_{ij}} \longrightarrow \tau_{U_{ij}}$$

$$\left(z, \frac{\partial}{\partial z_k}\right) \longmapsto \left(z, \sum_{\ell} \frac{\partial w_\ell}{\partial z_k} \frac{\partial}{\partial w_\ell}\right) \quad \leftarrow \text{chain rule}$$

$$\tilde{\psi}_{ij} = J(\varphi_{ij}) = \left(\frac{\partial w_\ell}{\partial z_k}\right)_{\ell,k}$$? transition functions of τ_X

ex: (the tautological line bundle)
v.b of rk 1

$\mathcal{O}(1)$ on $\mathbb{P}^n$

ex: (holomorphic cotangent bundle)

$\Omega_X := \tau_X^\vee$ = dual of the tangent bundle =
↑ there is this notion and it consists
on taking the dual locally
= bundle of holo 1-forms

ex: $\bigwedge^p \Omega_X =: \Omega_X^p$ bundle of holo p-forms $\left(\text{rk} \binom{n}{p}\right)$

$K_X := \bigwedge^n \Omega_X$ canonical (line) bundle.

- Sheaf of sections

$$\begin{array}{c} \text{vb } E \\ \downarrow \pi \\ X \end{array} \longleftrightarrow \text{sheaf of sections } E : \begin{array}{c} U \\ \cap \\ X \end{array} \longrightarrow E(U)$$
(sometimes $\mathcal{E}$)

Def: Let E be a v.b. on X and $U \subseteq X$ open. Then

$$E(U) = \left\{ \begin{tikzcd} & E \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \subseteq & X \end{tikzcd} \right\} = \left\{ U \xrightarrow{s} E \mid \pi \circ s = id \right\}$$

$$s \in E(U) \longmapsto s|_V \in E(V) \quad \forall V \subseteq U$$

EX: This is a sheaf (quite immediate because sections are functions)

ex: $E = X \times \mathbb{C}$ trivial line bundle $\Rightarrow E(U) = \left\{ \begin{tikzcd} & X \times \mathbb{C} \arrow[d, "pr_1"] \\ U \arrow[ur, "s"] \subset & X \end{tikzcd} \right\}$

$$\begin{tikzcd} X \times \mathbb{C} \arrow[d, "pr_1"] \\ X \end{tikzcd}$$

$s(x) = (x, f(x))$

where $f(x) = (pr_1 \circ s)(x)$ holo fun.

$f: U \longrightarrow \mathbb{C}$

- $\Rightarrow E(U) =$ holo fun. $U \longrightarrow \mathbb{C} \Rightarrow E = \mathcal{O}_X$

$E(x) = \pi^{-1}(x) = \left\{ \begin{tikzcd} & E \arrow[d, "\pi"] \\ \{x\} \arrow[ur, "s"] \arrow[r] & X \end{tikzcd} \right\}$ ← for one point the notation is consistent

- $E(x)$ is a $\mathbb{C}$-v.s.

$E(U)$ is a $\mathcal{O}_X(U)$-module:

- $s, t \in E(U) \Rightarrow s + t \in E(U)$: $(s+t)(x) = s(x) + t(x)$
- $f \in \mathcal{O}_X(U)$, $s \in E(U) \Rightarrow fs \in E(U)$: $(fs)(x) = f(x)s(x)$

ex: $E = X \times \mathbb{C}^r$ sheaf of sections

$$\begin{tikzcd} X \times \mathbb{C}^r \arrow[d] \\ X \arrow[u, bend left, "e_i"] \end{tikzcd} \qquad \mathcal{O}_X^{\oplus r} = \mathcal{O}_X \langle e_1, \ldots e_r \rangle \quad \text{free } \mathcal{O}_X\text{-module of rank } r$$

- $E(U) = \left\{ \sum f_i e_i \mid f_i \in \mathcal{O}_X(U) \right\}$

ex: $\mathcal{T}_X \cong X \times \mathbb{C}^n$, $\mathcal{T}_X(U) = \left\{ \sum f_i \frac{\partial}{\partial z_i} \right\}$, $X \subseteq \mathbb{C}^n$

- intrinsic def of sheaf $\mathcal{T}_X$:

$$\mathcal{T}_X(U) = Der_{\mathbb{C}}(U) = \left\{ \partial : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U) \mid \partial(fg) = f\partial(g) + g\partial(f) \right\}$$

$$\Omega_X(U) = \left\{ \sum f_i dz_i \right\}$$ Def the same way

Def: An $\mathcal{O}_X$-module $\mathcal{F}$ is locally free of rank r if $\exists X = \bigcup U_i$ open cover s.t $\mathcal{F}|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}_{U_i}^{\oplus r}$ isom. of $\mathcal{O}_{U_i}$-modules $\forall i$.

Prop: $\left\{ \text{vb } \begin{tikzcd} E \arrow[d] \\ X \end{tikzcd} \text{ of rk } r \right\} \Big/ \text{iso} \longleftrightarrow \left\{ \text{locally free } \mathcal{O}_X\text{-modules of rank } r \right\} \Big/ \text{iso}$

$$\left\{ \tilde{\varphi}_{ij} \text{ cocycles} \right\} \Big/ \sim$$

$$\left(\mathcal{F},\ \mathcal{F}|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}_{U_i}^{\oplus r}\right) \rightsquigarrow \psi_{ij} = \varphi_i \circ \varphi_j^{-1} : \mathcal{O}_{U_{ij}}^{\oplus r} \longrightarrow \mathcal{O}_{U_{ij}}^{\oplus r} \Longleftrightarrow$$
$$\Longleftrightarrow \tilde{\psi}_{ij} \in GL_r(\mathcal{O}_{U_{ij}}(U_{ij})) \Longleftrightarrow \tilde{\psi}_{ij} : U_{ij} \longrightarrow GL_r(\mathbb{C})$$

Operations on v.b:

E, E' v.b of rank r, r', with trans. fun ψ_{ij}, ψ'_{ij}

- $E \oplus E'$ rk $r+r'$, trans. fun: $\begin{pmatrix} \tilde{\psi}_{ij} & 0 \\ 0 & \tilde{\psi}'_{ij} \end{pmatrix}$
- $E \otimes E'$ rk rr', trans. fun. $\tilde{\psi}_{ij} \otimes \tilde{\psi}'_{ij}$
- $\bigwedge^p E$ rk $\binom{r}{p}$ trans fun. $\bigwedge^p \tilde{\psi}_{ij}$ special case
- $\det E$ rk 1 trans. fun. $\det(\tilde{\psi}_{ij})$ $\det(E) := \bigwedge^r E$
- $E^\vee$ rk r trans. fun. $(\tilde{\psi}_{ij}^T)^{-1}$

The key observation is that we have to ~~make~~ give sense to this operations in linear algebra sense.

- Picard group

L, L' line bundles $\Rightarrow L \otimes L'$ line bundle

$\tilde{\psi}_{ij} : U_{ij} \longrightarrow GL_1 = \mathbb{C}^\times$, ψ_{ij}, ψ'_{ij}

ψ_{ij} non-vanishing functions

$\Rightarrow L^\vee$ line bundle, ψ_{ij}^{-1}

$\Rightarrow L \otimes L^\vee = \mathcal{O}$ trivial v.b.

$\mathrm{Pic}(X) = \{\text{l.b. on } X\}/_{iso}$ with $\otimes$ as operation

is an abelian group

$L^{-1} := L^\vee$.

$\mathrm{Pic}(X) = \check{H}^1(X, GL_1) \circleddash$: Fact $= H^1(X, \mathcal{O}_X^\times)$ sheaf cohom.

$GL_1 = \mathcal{O}_X^\times$ = sheaf of non-vanishing holo fun.

$\circleddash \varinjlim_{X = \bigcup U_i} \check{H}^1(\{U_i\}, \mathcal{O}_X^\times)$ Čech cohom. Remember that $\varinjlim$ is the direct limit over ...

- Exponential sequence

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\exp} & \mathcal{O}_X^\times \\ f & \longmapsto & \exp\circ f \end{array}$$

$e^z = 1 \iff z = 2\pi i m,\ m \in \mathbb{Z}$, in the same way:

$$\exp\circ f = 1 \iff f(z) = 2\pi i m(z) \underset{X \text{ connected}}{\Longrightarrow} f(z) = 2\pi i m,\ m\in\mathbb{Z}$$

$$\ker(\exp)(U) = \ker(\exp(U)) \underset{U \text{ conn.}}{=} 2\pi i\mathbb{Z} \quad (*)$$

<u>Def</u>: A ab group. The constant sheaf with value A, sometimes denoted $\underline{A}$ or $\underline{A}_X$, is the sheaf

- $\underline{A}(U) = \mathrm{Map}_{\mathrm{cont}}(U, \underset{\text{discrete top.}}{A}) =$ locally const. maps $U \to A$ (const if U connected ?)

- Notice that we have the exact sequence: because of $(*)$

$$0 \longrightarrow 2\pi i\underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

hence exp is surjective (in the sheaf meaning!)
i.e., given $g \in \mathcal{O}_X^\times(U)$ (a non-vanishing holo func. on U),
$\exists\ U = \bigcup U_i$ open cover and logarithms $f_i = \log g|_{U_i}$

The idea of cohomology is to take short e.s. and give long e.s.:

- $0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^\times) \xrightarrow{\delta}$

$\hookrightarrow H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \boxed{H^1(X,\mathcal{O}_X^\times)} \xrightarrow{\delta}$, $H^1(X,\mathcal{O}_X^\times) = \mathrm{Pic}(X)$

- $\hookrightarrow H^2(X,\mathbb{Z})$

We are interested in studying $\mathrm{Pic}(X)$, so we can use the boundary map δ to $H^2(X,\mathbb{Z})$.

<u>Def</u>: The 1st Chern class of the l.b. L is $c_1(L) := \partial[L] \in$
$\in H^2(X,\mathbb{Z}) = H^2_{sing}(X,\mathbb{Z}) \longrightarrow H^2_{dR}(X,\mathbb{R})$

ex: Let $\underset{\dim m}{Y} \subseteq \underset{\dim n}{X}$ be a submanifold

$$\begin{array}{ccc} TX & & TX|_Y = \pi^{-1}(Y) \\ \downarrow \pi & \rightsquigarrow & \downarrow \\ X & & Y \end{array} \quad \text{v.b. on } Y$$

$$\begin{tikzcd} TY \arrow[rr, "f"] \arrow[dr] & & TX|_Y \arrow[dl] \\ & Y & \end{tikzcd}$$ f is a homom. of v.b.

Def: $E \xrightarrow{f} F$ is a homomorphism of v.b. if

- f holo and
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & \circlearrowleft & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$
- $f(x)$ $\mathbb{C}$-linear on $E(x) \to F(x)$
- rk $f(x)$ is constant (varying x)

Note: constant rank $\Rightarrow$ $\ker(f)$, $\mathrm{im}(f)$, $\mathrm{coker}(f)$ v.b.

In the previous example:

$$0 \longrightarrow T_Y \xrightarrow{f} T_X|_Y \longrightarrow \mathrm{coker}(f) \longrightarrow 0 \quad \text{e.s. of v.b}$$

$\mathrm{coker}(f) = N_{Y/X}$ normal bundle of Y in X.
(we don't have the notion of normality $\perp$)

$\hat{\imath}$ does not split in general but every e.s. of v.b splits locally.