splits locally. (... normality $\perp$)

- Line bundle on $\mathbb{P}^n$.

10/03/20

– Tautological line bundle $\mathcal{O}(-1)$

<u>Def</u>:

$$\begin{tikzcd} \mathcal{O}(-1) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\pi"', "\text{sub bundle}"] & \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[d, "\text{trivial v.b of rank } n+1"] \\ & \mathbb{P}^n \end{tikzcd}$$

$\mathbb{P}^n = \{ \ell \subset \mathbb{C}^{n+1} \text{ lines, i.e. linear subsp. of dim } 1 \} =$
$= \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^\times$

$\mathcal{O}(-1) = \{ (\ell, z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid z \in \ell \}$ as a set.

$\pi^{-1}(\ell) = \{ z \in \ell \} \cong \mathbb{C}$ linear 1-dim. space

Verify it's a line bundle:

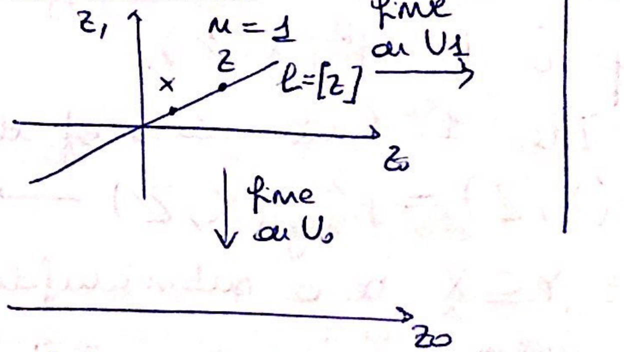
$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i , \quad U_i = \{ (z_0 : \_ : z_n) \mid z_i \neq 0 \} \cong \mathbb{C}^n$

$\varphi_i : \mathcal{O}(-1)|_{U_i} \longrightarrow U_i \times \mathbb{C}$ ← we have to find such a trivialization

$\mathcal{O}(-1)|_{U_i} = \pi^{-1}(U_i)$

$(\ell, x) \longmapsto (\ell, x_i)$

$(\ell, z \frac{w}{z_i}) \longleftarrow (\ell, w)$



Let $\mathcal{O}(-1)$ as a manifold be given by φ_i.

Remember that $\varphi_i|_x$ is linear $\Longrightarrow$ $\mathcal{O}(-1)$ is a line bundle.

Transition functions: $\psi_{ij} = \psi_i \circ \psi_j^{-1} : U_{ij} \times \mathbb{C} \longrightarrow U_{ij} \times \mathbb{C}$
$$(\ell, w) \longmapsto \left(\ell, \left(z \frac{w}{z_j}\right)_i\right) = \left(\ell, \frac{z_i}{z_j} w\right)$$

we can also think it as:
$$\psi_{ij} : U_{ij} \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$
$$z \longmapsto \frac{z_i}{z_j}$$

Def: $\mathcal{O}(1) = \mathcal{O}(-1)^\vee = \mathcal{O}(-1)^{-1}$ has trans. functions:
$$\psi_{ij} = \frac{z_j}{z_i}$$

Def: $\mathcal{O}(k) = \begin{cases} \mathcal{O}(1)^{\otimes k} & k>0 \\ \mathcal{O}(-1)^{\otimes k} & k<0 \\ \mathcal{O} & k=0 \end{cases}$

NB: $\otimes$ is just defined by the multiplication

- where $\mathcal{O} = \mathbb{P}^n \times \mathbb{C} = \mathcal{O}_{\mathbb{P}^n}$ trivial line bundle.
- $\mathcal{O}(k)$ has trans. functions $\psi_{ij} = \left(\frac{z_j}{z_i}\right)^k$

Rem: $\mathcal{O}(k)$ can be constructed in different ways:
$$\begin{tikzcd} (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} \arrow[r, "/\mathbb{C}^\times"] \arrow[d, "\text{trivial line bundle}"'] & \mathcal{O}(k) \arrow[d] \\ \mathbb{C}^{n+1} \setminus \{0\} \arrow[r, "/\mathbb{C}^\times"] & \mathbb{P}^n \end{tikzcd}$$

where $\mathbb{C}^\times$-action on $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$, $\lambda \in \mathbb{C}^\times$, $(z, w) \in \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$, is $\lambda(z,w) = (\lambda z, \lambda^k w)$

EX: Verify that this gives the same line bundle as before.

ex: $\mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus n+1} = \mathbb{P}^n \times \mathbb{C}^{n+1}$ (as v.b.)

NB: E, F v.bs with trans. functs ψ_{ij} and φ_{ij} resp.
$\Rightarrow$ $E \oplus F$ is a v.b with trans. f. $\begin{pmatrix} \psi_{ij} & 0 \\ 0 & \varphi_{ij} \end{pmatrix}$

- this map is given by
- $$\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} \longrightarrow \mathcal{O}(-1) \dashrightarrow \mathbb{P}^n \times \mathbb{C}^{n+1}$$
$$(z, w) \longmapsto [(z,w)] \longmapsto ([z], z_0 w, z_1 w, \ldots, z_n w)$$

Similarly:
$$\mathcal{O}(-k) \longrightarrow \mathcal{O}$$
$$(z,w) \longmapsto (z, f(z) w)$$
is well defined if f homogeneous of deg k.

So we have:
$$\mathcal{O}(-1) \xrightarrow{z_i} \mathcal{O}$$
$$\mathcal{O}(-k) \xrightarrow{f} \mathcal{O} \qquad f \text{ hom. of deg } k$$

which give by taking duals:
$$\mathcal{O} \xrightarrow{z_i} \mathcal{O}(-1)^\vee = \mathcal{O}(1)$$
$$\mathcal{O} \xrightarrow{f} \mathcal{O}(-k)^\vee = \mathcal{O}(k)$$

called in the same way as before because in local coordinates they look the same.

Rem: Homomorphisms $\mathcal{O} \longrightarrow E$ $\quad (E \text{ v.b})$

$\Longleftrightarrow$ sections $s \in \Gamma(X, E)$

Pf

$$\begin{tikzcd} X \times \mathbb{C} \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

$\varphi(x) : \mathbb{C} \longrightarrow \underset{E(x)}{\overset{}{\mathbb{C}^r}}$ linear $\overset{!}{\Longleftrightarrow}$ element of $\mathbb{C}^r$

$1 \longmapsto \varphi(x)(1)$

$s(x) := \varphi(x)(1)$ so $\varphi(x)$ ---

verify φ holo $\Longleftrightarrow$ s holo.

So polynomials homogeneous of deg k gives sections of $\mathcal{O}(k)$.

"twisted holomorphic functions".

Prop: $\Gamma(\mathbb{P}^n, \mathcal{O}(k)) = \mathbb{C}[z_0, \ldots z_n]_k$ = hom. pol of deg k.

"Pf" Pick a section of $\mathcal{O}(k)$ $\Longleftrightarrow$ equivariant ($\mathbb{C}^\times$- compatible) section t

$$\begin{tikzcd} (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} \arrow[r] \arrow[d, shift left] & \mathcal{O}(k) \arrow[d, "\pi"] \\ \mathbb{C}^{n+1} \setminus \{0\} \arrow[r] \arrow[u, bend left, "t"] & \mathbb{P}^n \arrow[u, bend left, "s"] \end{tikzcd}$$

$\Updownarrow$

holo fun $t : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}$

such that $t(\lambda z) = \lambda^k t(z)$

is a "twisted holo fun".

Th (Hartogs):

$\Longrightarrow$ t extends to a holo fun $\mathbb{C}^{n+1} \cancel{\setminus \{0\}} \longrightarrow \mathbb{C}$

($n \geq 1 \Rightarrow 0 \subset \mathbb{C}^{n+1}$ codim $n+1 \geq 2$)

Look at germ $t \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$, $t(z) = \sum_{\alpha \in \mathbb{N}^{n+1}} c_\alpha z^\alpha$ convergent power series in a nbhd of 0.

$t(\lambda z) = \sum c_\alpha \lambda^{|\alpha|} z^\alpha = \lambda^k t(z) \Rightarrow |\alpha| = k$ if $c_\alpha \neq 0$.

. Canonical bundle on $\mathbb{P}^n$

Recall: X cplex manifold, T_X tangent bundle, $\Omega_X = T_X^\vee$ cotangent bdl (rank $n = \dim X$)

Canonical line bundle: $K_X = \det(\Omega_X)$

Locally: $K_X = \mathcal{O}_X \, dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n$

$X \subset \mathbb{C}^n$ open

Prop: $K_{\mathbb{P}^m} \cong \mathcal{O}(-m-1)$

Pf (one of the possible)

Write down transition matrices $\psi_{ij} : U_{ij} \to GL_m(\mathbb{C})$. Calculate $\det(\psi_{ij})$ and match with $\mathcal{O}(-m-1)$.

(another possibility, more conceptual)

$\exists$ Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^m} \xrightarrow{\alpha} \bigoplus_{i=0}^{m} \mathcal{O}(-1) \xrightarrow{\beta} \mathcal{O} \longrightarrow 0$$

$$w_0, \dots, w_m \longmapsto z_0 w_0 + z_1 w_1 + \dots + z_m w_m \quad (*)$$

is exact (kernel of β is the cotangent bundle)

Dually:

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_{i=0}^{m} \mathcal{O}(1) \longrightarrow T_{\mathbb{P}^m} \longrightarrow 0$$

From $(*)$. Locally we have charts where transition fun of $\bigoplus_{i=0}^{m}\mathcal{O}(-1)$ look like (this works for any exact sequence)

$$\left(\begin{array}{c|c} \psi_{ij}^{\Omega_{\mathbb{P}^m}} & 0 \\ \hline 0 & \psi_{ij}^{\mathcal{O}} \end{array}\right)$$

$$\rightsquigarrow \det\left(\bigoplus_{i=0}^{m}\mathcal{O}(-1)\right) \cong \det(\Omega_{\mathbb{P}^m}) \otimes \underbrace{\det(\mathcal{O})}_{=\mathcal{O}} = K_{\mathbb{P}^m}$$

trans fun: $\begin{pmatrix} \psi_{ij}^{\mathcal{O}(-1)} & & 0 \\ & \ddots & \\ 0 & & \psi_{ij}^{\mathcal{O}(-1)} \end{pmatrix}$

$$\text{LHS} \cong \mathcal{O}((m+1)(-1)) = \mathcal{O}(-m-1)$$

(left hand side)

To find an Euler exact sequence we have to work over $U_i \subseteq \mathbb{P}^m$. $U_i \cong \mathbb{C}^m$, coord. $\frac{z_k}{z_i}$

$\Omega_{\mathbb{P}^m}$ is a trivial v.b with basis $d\left(\frac{z_k}{z_i}\right)$, $k \neq i$

($\mathcal{O}(-1)$ is a trivial v.b with basis $\{z_i\}$)

$\bigoplus_{i=0}^{m}\mathcal{O}(-1)$ is a trivial v.b with basis $e_0, \dots, e_m$.

β becomes $\left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \dots \frac{z_m}{z_i}\right)$, $\ker(\beta) = \left\langle \frac{z_k}{z_i} e_i - e_k \right\rangle_{k \neq i}$

$\alpha_i\left(d\left(\frac{z_k}{z_i}\right)\right) = e_k - \frac{z_k}{z_i} e_i$ (α_i is α on U_i)

How does ith chart cooperate with j-th chart?

$$d\left(\frac{f}{g}\right) = d(f)\frac{1}{g} + f\, d\left(\frac{1}{g}\right) = d(f)\frac{1}{g} + f\left(-\frac{1}{g^2}\right) d(g)$$

$$\Rightarrow d\left(\frac{z_k}{z_i}\right) = d\left(\frac{z_k}{z_j} \Big/ \frac{z_i}{z_j}\right) = d\left(\frac{z_k}{z_j}\right)\frac{z_j}{z_i} + \frac{z_k}{z_j}\left(-\frac{1}{(z_i/z_j)^2}\, d\left(\frac{z_i}{z_j}\right)\right) =$$

$$= \frac{z_j}{z_i}\left(\overbrace{d\left(\frac{z_k}{z_j}\right)}^{\star} - \overbrace{\frac{z_k}{z_i}\, d\left(\frac{z_i}{z_j}\right)}^{\heartsuit}\right)$$

over U_{ij}:

$$\alpha_j\left(d\left(\frac{z_k}{z_i}\right)\right) = \frac{z_j}{z_i}\Big(\underbrace{e_k - \frac{z_k}{z_j}e_j}_{*} - \underbrace{\frac{z_k}{z_i}\left(e_i - \frac{z_i}{z_j}e_j\right)}_{\heartsuit}\Big) =$$

$$= \frac{z_j}{z_i}\left(e_k - \frac{z_k}{z_j}e_i\right) = \frac{z_j}{z_i}\alpha_i\left(d\left(\frac{z_k}{z_i}\right)\right)$$

Fact: $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$
$\qquad \mathcal{O}(k) \longleftrightarrow k$

Why? We have seen the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\exp} \mathcal{O}^\times_{\mathbb{P}^n} \longrightarrow 1$$

cohomology gives long exact sequence

$$0 \to H^0(\mathbb{P}^n, \mathbb{Z}) \to \underbrace{H^0(\mathbb{P}^n, \mathcal{O})}_{=\mathbb{C}} \to H^0(\mathbb{P}^n, \mathcal{O}^\times) \to$$

$$\to H^1(\mathbb{P}^n, \mathbb{Z}) \to \underbrace{H^1(\mathbb{P}^n, \mathcal{O})}_{=0} \to H^1(\mathbb{P}^n, \mathcal{O}^\times) = \mathrm{Pic}(\mathbb{P}^n) \to$$

$$\to \underbrace{H^2(\mathbb{P}^n, \mathbb{Z})}_{=\mathbb{Z}} \to \underbrace{H^2(\mathbb{P}^n, \mathcal{O})}_{=0}$$

all claims! Must be proved

$$\Rightarrow \quad 0 \to \mathrm{Pic}(\mathbb{P}^n) \xrightarrow{\cong} \mathbb{Z} \to 0$$

By elementary topology considerations:

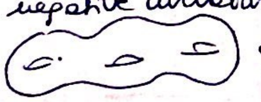
$$H^2_{sing}(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 4, \dots 2n \\ 0 & \text{otherwise} \end{cases}$$

and it can be shown:

$$H^i(\mathbb{P}^n, \mathcal{O}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i > 0 \end{cases}$$

ex: $X = \mathbb{C}^g/\Lambda$, $\Omega_X = \mathcal{O}_X\langle dz_1, \dots dz_g\rangle$ with $d(z+\lambda) = dz$, $\lambda \in \Lambda$

$K_X = \mathcal{O}$ trivial l. b.

				Ko. dim
Note: $\mathbb{P}^1$: pos. curvature 	$g = 0$,	$K_{\mathbb{P}^1} = \mathcal{O}(-2)$,	no sections	$k = -\infty$
$\mathbb{C}/\Lambda$ flat 	$g = 1$,	$K_{\mathbb{C}/\Lambda} = \mathcal{O}$,	sections $\mathbb{C}$	$k = 0$
negative curvature 	$g > 1$	$K = ?$,	many sections	$k = 1$