Complex Algebraic Geometry
Lecture #7, Mar 17, 2020

- Analytic subvarieties
- Analytic hypersurfaces
- Divisors
  - Weil divisors
  - Principal divisors
  - Cartier divisors
  - $\mathrm{CDiv} \cong \mathrm{WDiv}$

Analytic subvarieties/subsets

X complex manifold.

Def: An analytic subset = subvariety is a closed subset $Z \subset X$ s.th., locally on X, $Z = Z(f_1, \dots, f_k)$ f_i holo.

Def: $Z_{reg} = \{x \in X : \exists n, g_1, \dots, g_n :$ loc $Z = Z(g_1, \dots, g_n)$ a manifold $\quad x$ reg point of the map $X \xrightarrow{(g_i)} \mathbb{C}^n \}$

open subset

Def: $Z_{sing} = Z \setminus Z_{reg}$.

Def: $I_Z \subset \mathcal{O}_X$ ideal sheaf of Z $\qquad \left(0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0\right)$

$I_Z(U) = \{ f \in \mathcal{O}_X(U) : f(z) = 0 \ \forall z \in Z \}$

holo fcns on Z if Z manifold

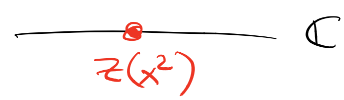
Rmk: $Z = Z(f_1, \dots, f_k)$ $\quad f_i \in I_Z(X)$ $\quad (f_1, \dots, f_k) = \{\sum f_i h_i\} \subset I_Z(X)$

can be strict!

Thm (Nullstellensatz 1.1.29) $I_Z = \sqrt{(f_1, \dots, f_k)}$

$I_Z(U) = \{ f \in \mathcal{O}_X(U) : \exists N : f^N \in (f_1, \dots, f_k) \}$

Ex: $Z = Z(x^2) \subset \mathbb{C}^1$ $\quad I_Z = (x)$



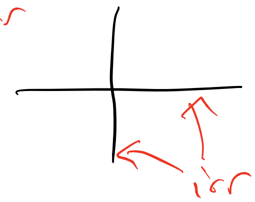
Def: Z irreducible if $\nexists\, Z = Z_1 \cup Z_2$ where $Z_i \subset X$ analytic subsets.

Z loc irr at $z \in Z$ if $\forall U \ni z$ nbhds $Z \cap U$ irreducible.

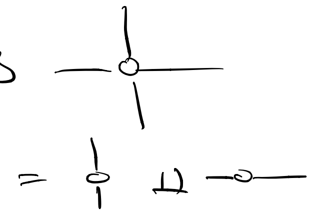
Lem 1.1.28: Z irr at $z \iff I_{Z,z} \subset \mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n, 0}$ prime ideal

Ex: 1) $Z = Z(xy) \subset \mathbb{C}^2$ not irr

$= Z(x) \cup Z(y)$
$\quad$ irr $\quad$ irr

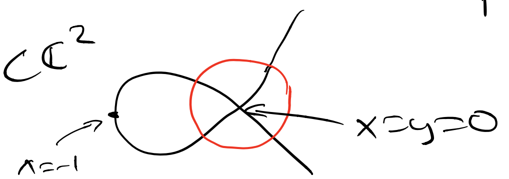


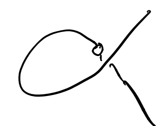
Z_{reg}



2) $Z = Z(y^2 - x^3 - x^2) \subset \mathbb{C}^2$

outside $x = -1$





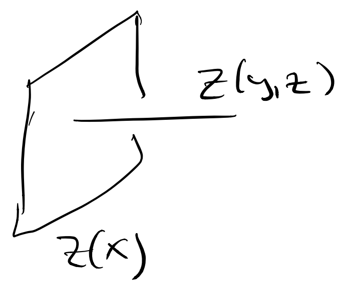
$y^2 - x^3 - x^2 = (y - x\sqrt{1+x})(y + x\sqrt{1+x})$

Z irreducible, not loc irr at 0.

3) $Z = Z(xy, xz)$

$= Z(x) \cup Z(y,z)$

dim 2 $\quad$ dim 1
codim 1 $\quad$ codim 2



Facts:
- $Z_{reg} \subset Z$. If Z irr $\Rightarrow Z_{reg}$ connected
- (easy) locally write $Z = \bigcup_{i=1}^{n} Z_i$ w/ Z_i irreducible.
- globally write $Z = \bigcup_{\alpha} Z_\alpha$ w/ Z_α irr. (loc. finite)
- X compact ($\Rightarrow Z$ compact) $\Rightarrow Z = \bigcup_{i=1}^{n} Z_i$ (not obvious)
- $Z_{reg} = \coprod \underbrace{(Z_i \cap Z_{reg})}_{\text{conn.}}$

Def: $\dim Z = \dim Z_{reg}$ if Z irreducible.
$\dim Z = \max \dim Z_\alpha$ if $Z = \bigcup Z_\alpha$

Fact: • Z irr, then $\dim Z = \dim(\mathcal{O}_{X,x}/\mathcal{I}_{Z,x})$ Krull dim.

• $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n,0}$ noetherian UFD

$\underset{\substack{\text{UFD} \\ \text{(normality)}}}{\Longrightarrow}$ if Z irr has codim 1, then $\mathcal{I}_{Z,x} = (f)$ $f \neq 0$
(not true in higher dim)
$\Rightarrow Z = Z(f)$ locally at x

Analytic hypersurfaces

Def: An analytic hypersurface Z is an analytic subvariety of codim 1
$\Leftrightarrow$ locally $Z = Z(f)$

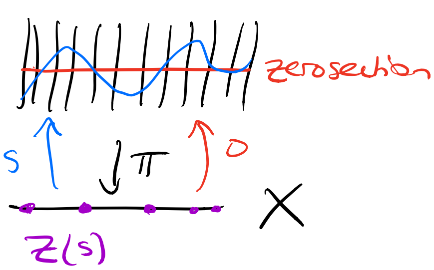
Ex: $p(z_0, \dots, z_n)$ hom pol of deg d $\rightsquigarrow Z(p) \subset \mathbb{P}^n$

Locally given by $Z(p_i) \subset U_i$ $\quad p_i(x_0, \dots, \check{x}_i, \dots, x_n) = p\left(\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}\right)$

$Z(p)$ is an analytic hypersurface.

Def: Let L be a line bundle on X and $s \in \Gamma(X, L) = L(X)$ be a global section. $Z(s) = s^{-1}(0) \subset X$ zero locus of s

$$\begin{tikzcd} L \arrow[d, "\pi"] \\ X \arrow[u, bend left=60, "s"] \arrow[u, bend right=60, "0"'] \end{tikzcd}$$



Rmk: Assume $s \not\equiv 0$ on every conn. comp of X. Then $s^{-1}(0)$ analytic hypersurface b/c locally:

$$\begin{tikzcd} L \cong X \times \mathbb{C} \arrow[d, "pr_1"] \\ X \arrow[u, bend left, "s"] \arrow[u, bend right, "s"'] \end{tikzcd}$$

$s \longleftrightarrow$ holo fcn $f: X \to \mathbb{C}$

$s(x) = (x, f(x))$ $\qquad$ $f(x) = pr_2(s(x))$

$s^{-1}(0) = f^{-1}(0)$

Ex (revisited) p hom pol of deg d $\longleftrightarrow$ $s \in \Gamma(X, \mathcal{O}(d))$

$Z(p) \qquad = \qquad Z(s)$

Want: to remember mult. of $Z(s)$.

Ex: $Z(x^2) = Z(x)$ $\qquad$ Answer: Divisors.

(Sidenote: If $E \to X$ vector bundle of rank r, $s \in E(X)$ global section then $s^{-1}(0)$ anal. subvariety of codim $\leq r$.

However, not every subvariety of codim $\leq r$ arise in this way, not even for submanifolds. (and we'll see that for $r=1$ they do))

Divisors

Def: A (Weil) divisor D on X is a formal linear comb

should be locally finite

$$D = \sum a_i [Z_i] \qquad a_i \in \mathbb{Z} \text{ (multiplicities)}$$
$$Z_i \subset X \text{ irr. anal. hypersurf}$$

= anal. subset of codim 1.

Notation: If $Y = \cup Y_i$ anal hypersurf, then

$$[Y] := \sum [Y_i]$$

Ex: $[Z(xy)] = [Z(x)] + [Z(y)]$

Def: $\mathrm{WDiv}\, X = \mathrm{Div}\, X$ group of (Weil) divisors

(free abelian group on irr. anal. hypersurf)

Def: $\mathrm{Supp}(D) = \cup Z_i \quad (D = \sum a_i [Z_i] \quad a_i \neq 0)$

analytic hypersurf

Def: D is effective if $a_i \geq 0 \ \forall i$

Principal divisors

Rmk: Holomorphic fcn $\longrightarrow$ hypersurf $\longrightarrow$ effective divisor

like bdl + section

Def: $f \neq 0$ meromorphic function. Associated divisor:

$$\mathrm{div}(f) = (f) := Z(f) - P(f) = \sum_{Y \text{ irr hypersurf}} \mathrm{ord}_Y(f)\,[Y] \in \mathrm{Div}X$$

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Y irreducible hypersurface, $f \in K(Y)$

Def: $\operatorname{ord}_Y(f) := \operatorname{ord}_{Y,y}(f)$ for some $y \in Y_{reg}$ such general. e.g. $y \in (Y \cap \operatorname{Supp} f)_{reg}$

$\operatorname{ord}_{Y,y}(f)$ defined by $f = \frac{f_1}{f_2} = g^{\operatorname{ord}} \cdot h$ $I_{Y,y} = (g)$, $h \in \mathcal{O}^\times_{X,y}$

well defined b/c $\mathcal{O}_{X,y}$ is an UFD.

(g only defined up to mult w/ elt in $\mathcal{O}^\times_{X,y}$)

(Check: independent on the choice of y)

prob1.
Ex: $Y = Z(x)$ $\mathbb{C}^2$, $g = x$
$f = \frac{1}{y} \in K(Y)$

$\operatorname{ord}_Y(f) = \operatorname{ord}_{Y,0}(f)$

$\nexists\, f = x^{\operatorname{ord}} \cdot h$

Sol 1: Pick $y \neq 0$ Sol 2: $\operatorname{ord}_Y\left(\frac{f}{g}\right) := \operatorname{ord} f - \operatorname{ord} g$

$\operatorname{ord}_Y(f) = 0$

poles of f, y, x, Y

Rmk: $\operatorname{div}\left(\frac{f_1}{f_2}\right) = \operatorname{div}(f_1) - \operatorname{div}(f_2)$

$\operatorname{div}(f_1 f_2) = \operatorname{div}(f_1) + \operatorname{div}(f_2)$

(If $Y \not\subset \operatorname{Supp}(f) \Leftrightarrow \operatorname{ord}_Y(f) = 0$)

Thus, have a group homomorphism $K(X)^\times \xrightarrow{\varphi} \operatorname{Div} X$.

Def: $\operatorname{PDiv} X = \operatorname{im} \varphi = \{D = \operatorname{div}(f) : f \text{ meromorphic}\}$

group of principal divisors

$\mathcal{O}_X(U)^\times \subset \mathcal{O}_X(U)$

$\mathcal{O}^\times_X(U) = \{f \in \mathcal{O}_X(U) : f(z) \neq 0\ \forall z \in U\}$

$K^\times_X(U) = K(U)^\times = K(U) \setminus 0$ if U connected

Cartier divisors

"locally meromorphic function"

Exact seq: $1 \to \mathcal{O}^\times_X \to K^\times_X \to \boxed{K^\times_X / \mathcal{O}^\times_X} \to 1$

quotient sheaf

Recall:

$1 \to \mathcal{O}^\times_X(X) \to K(X)^\times \to (K^\times_X / \mathcal{O}^\times_X)(X)$

$K(X)^\times / \mathcal{O}^\times_X(X)$

not necc surj! (i.e. $\operatorname{PDiv} \neq \operatorname{CDiv}$)

But locally a section of $K^\times_X / \mathcal{O}^\times_X$ arises from a section of $K^\times_X$.

<u>Def</u>: $\mathrm{CDiv}\, X = \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ group of Cartier divisors.

<u>Rmk</u>: $\mathrm{PDiv}\, X \longrightarrow \mathrm{CDiv}\, X$.

<u>Prop 2.3.9</u>: $\mathrm{CDiv}\, X \cong \mathrm{WDiv}\, X$ (needs X manifold)

From LHS $\to$ RHS: $s \in (\mathcal{K}_X^\times/\mathcal{O}_X^\times)(X)$

unique up to mult $h \in \mathcal{O}_{U_i}^\times$

Then $\exists X = \bigcup U_i$ and $f_i \in \mathcal{K}_X^\times(U_i)$ meromorphic s.th. $s|_{U_i} = \overline{f_i} \in (\mathcal{K}_X^\times/\mathcal{O}_X^\times)(U_i)$

$$s \longmapsto \underset{\text{mero on } U_i}{(f_i)} \longmapsto \underset{\mathrm{WDiv}(U_i)}{(\mathrm{div}(f_i))} \overset{\text{glue}}{\longmapsto} \underset{\mathrm{WDiv}\, X}{\mathrm{div}(s)}$$

welldef: $\mathrm{div}(f_i)|_{U_i \cap U_j} = \mathrm{div}(f_j)|_{U_i \cap U_j}$

$f_i|_{U_{ij}} = f_j|_{U_{ij}} \cdot h \quad h \in \mathcal{O}_{U_{ij}}^\times$ neither zeroes nor poles

$\Rightarrow \mathrm{div}(f_i)|_{U_{ij}} = \mathrm{div}(f_j)|_{U_{ij}}$

From RHS $\to$ LHS: $D = \sum a_k [Y_k]$ $\quad Y_k$ irr hypersurface

Then $\exists X = \bigcup U_i$ and $Y_k \cap U_i = Z(f_{ki})$ or more precisely $I_{Y_k} = (f_{ki})$ on U_i X mfld

unique up to mult by $h \in \mathcal{O}_{U_i}^\times$

$$D \longmapsto \underset{\text{mero on } U_i}{\left(\prod_k f_{ki}^{a_k}\right)} \longmapsto \underset{(\mathcal{K}_X^\times/\mathcal{O}_X^\times)(U_i)}{\overline{\prod_k f_{ki}^{a_k}}} \overset{\text{glue}}{\longmapsto} \underset{\mathcal{K}_X^\times/\mathcal{O}_X^\times(X)}{s}$$

<u>pf</u>: Verify that these 2 constructions are inverse to each other.

<u>Rmk</u>: $Y \subset X$ anal subvar of codim ≥ 2

$\Rightarrow \mathrm{WDiv}\, X = \mathrm{WDiv}(X \setminus Y)$

(uses that holo fcns extend: $\mathcal{O}_X(X) \xrightarrow{\cong} \mathcal{O}_X(X \setminus Y)$)

cf HW to lec #6

Generalizations:

$$\mathrm{WDiv} = \text{codim } 1 \text{ cycles} = \sum a_i [Y_i]$$

$$= Z^1(X)$$

Y_i irr hypersurf

$$Z^r(X) = \text{codim } r \text{ cycles} = \sum a_i [Y_i] \quad Y_i \text{ irr subvar of codim } r$$

$$Z_d(X) = \dim d \text{ cycle}$$

$$\dim Y_i = \underbrace{\dim X}_{=n} - \operatorname{codim} Y_i$$

X conn of dim n $\quad Z^r(X) = Z_{n-d}(X)$

Next time:

$$\mathrm{Div}\, X \longrightarrow \mathrm{Pic}\, X \overset{\text{e.g. } X \text{ proj}}{=} \mathrm{Div}\, X / \mathrm{PDiv}\, X$$

cycles $\qquad$ cycle classes

Chow groups