Complex Algebraic Geometry
Lecture #8, Mar 24, 2020
(SF3612)

- Divisors and line bundles $\mathcal{O}(D)$
- Linear equivalence, homological equiv.
- Sections of line bundles and divisors

Recall: $\mathrm{WDiv}\, X = \{ \sum a_i [Y_i] : a_i \in \mathbb{Z},\ Y_i \text{ hypersurface} \}$

$$\mathrm{CDiv}\, X = \{ (U_i, f_i) : f_i = f_j \cdot \psi_{ij} \} = \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$$

$$X = \cup U_i \quad \overset{\wedge}{K(U_i)^\times} \quad \psi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^\times)$$

$\mathrm{WDiv}\, X \xrightarrow{\sim} \mathrm{CDiv}\, X$ group isomorphism

$\mathrm{PDiv}\, X = \{ \mathrm{div}(f) : f \in \mathcal{K}_X^\times \}$

Divisors and line bundles

We want to define a group homo $\mathrm{Div}\, X \to \mathrm{Pic}\, X$:

$$D \in \mathrm{Div}\, X \longmapsto \mathcal{O}(D) \in \mathrm{Pic}\, X$$

$$D_1 + D_2 \longmapsto \mathcal{O}(D_1 + D_2) \cong \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)$$

To define $\mathcal{O}(D)$ we need to give $\varphi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*)$.

If $X = \bigcup U_i$ cover $D \leftrightarrow (U_i, f_i)$, then

$$\varphi_{ij} := f_i / f_j$$

Need to verify cocycle condition:

$$\underbrace{\varphi_{ij}}_{f_i/f_j} \cdot \underbrace{\varphi_{jk}}_{f_j/f_k} = \underbrace{\varphi_{ik}}_{f_i/f_k}$$

Ok.

ADDITION $D + D' \longleftrightarrow (U_i, f_i f_i') \longmapsto (U_{ij}, \varphi_{ij}\varphi_{ij}')$ $\quad \varphi_{ij}\varphi_{ij}' = \dfrac{f_i f_i'}{f_j f_j'}$

$$\mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$$

$$\varphi_{ij}\varphi_{ij}' = \varphi_{ij} \otimes \varphi_{ij}'$$

Rmk: $D = 0$ empty sum $= \mathbb{Z}(1)$

UNIT as Cartier div $(X, 1) \longmapsto \mathcal{O}(0) = (X, 1) = \mathcal{O}$

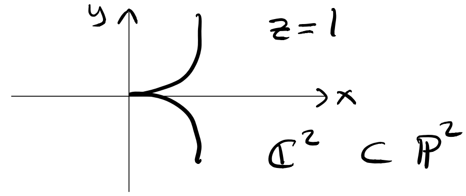
trivial line bundle $X \times \mathbb{C}$

INVERSE $-D \longleftrightarrow (U_i, f_i^{-1}) \longmapsto (U_{ij}, \varphi_{ij}^{-1}) = \mathcal{O}(D)^\vee$

Ex: $D - D \longmapsto \mathcal{O}(D) \otimes \mathcal{O}(D)^\vee = \mathcal{O}$

<u>Ex</u>: $f = y^2 z - x^3$ homogeneous of degree 3

$Z(f) \subset \mathbb{P}^2 = \{(x:y:z)\}$



On chart $U_x = \{x \neq 0\}$:

$$f_x = \frac{f}{x^3} = \left(\frac{y}{x}\right)^2 \frac{z}{x} - 1$$

On chart U_y: $f_y = \frac{f}{y^3} = \frac{z}{y} - \left(\frac{x}{y}\right)^3$

On chart U_z: $f_z = \frac{f}{z^3} = \left(\frac{y}{z}\right)^2 - \left(\frac{x}{z}\right)^3$

Gives divisor $\operatorname{div}(f) = \left\{ \left(U_x, \frac{f}{x^3}\right), \left(U_y, \frac{f}{y^3}\right), \left(U_z, \frac{f}{z^3}\right) \right\}$

Transition functions:

$$\psi_{xy} = \frac{f_x}{f_y} = \frac{y^3}{x^3}, \quad \psi_{yz} = \frac{z^3}{y^3}, \quad \psi_{zx} = \frac{x^3}{z^3}$$

$\psi_{xy} \in \Gamma(U_x \cap U_y, \mathcal{O}^{\times}_{\mathbb{P}^2})$

So $\mathcal{O}(\operatorname{div}(f)) \cong \mathcal{O}(3)$.

In general $\mathcal{O}(\operatorname{div}(\text{hom pol of deg } d)) \cong \mathcal{O}(d)$.

Linear equivalence

Lemma (2.3.14) $D \in \operatorname{Div} X$ is principal $\iff \mathcal{O}(D)$ trivial l.b.

pf: ($\Rightarrow$) $D = \operatorname{div}(f)$ f meromorphic.

Can take $X = X$ trivial cover $\rightsquigarrow$ trivial l.b.

or $X = \bigcup U_i$ $(X, f) = (U_i, f|_{U_i})_i \rightsquigarrow \psi_{ij} = \dfrac{(f|_{U_i})|_{U_{ij}}}{(f|_{U_j})|_{U_{ij}}} = 1$

($\Leftarrow$) $D = (U_i, f_i)$ $f_i \in K(U_i)^\times$, $\mathcal{O}(D)$ trivial $\iff \forall i \exists g_i \in \Gamma(U_i, \mathcal{O}_X^\times)$

$\psi_{ij} = f_i / f_j$ s.th. $\psi_{ij} = g_i / g_j$

$\tilde{f}_i := f_i / g_i$ Then $D = (U_i, \tilde{f}_i)$

The $\tilde{f}_i$ glue ($\tilde{f}_i|_{U_{ij}} = \tilde{f}_j|_{U_{ij}}$) to $\tilde{f} \in K(X)^\times$

$\Rightarrow D = \operatorname{div}(\tilde{f})$ principal. $\square$

Comment: $\mathcal{O}(D) \in \operatorname{Pic} X$ is the "obstruction" to D being principal.

Slightly different point of view:

S.E.S. $$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times / \mathcal{O}_X^\times \to 1$$

$\rightsquigarrow$ LES
$$1 \to \Gamma(X, \mathcal{O}_X^\times) \to \underset{\substack{\| \\ K(X)^\times}}{\Gamma(X, \mathcal{K}_X^\times)} \xrightarrow{\alpha} \underset{\substack{\| \\ \operatorname{Div} X}}{\Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)} \xrightarrow{\beta} \underset{\substack{\| \\ \operatorname{Pic} X}}{H^1(X, \mathcal{O}_X^\times)} \to$$

exact means: $\underset{\underset{\text{PDiv}}{\shortparallel}}{\operatorname{im}\alpha} = \ker\beta$

$\Rightarrow$ SES: $1 \longrightarrow \operatorname{PDiv} X \longrightarrow \operatorname{Div} X \xrightarrow{\beta} \operatorname{Pic} X$
$\qquad\qquad\qquad\qquad\qquad\qquad D \longmapsto \mathcal{O}(D)$

Rmk: β is not always surjective. It is surj when X projective.

Def: The class group of X is the image of β

$$\operatorname{Cl} X := \operatorname{Div} X / \operatorname{PDiv} X \qquad (\text{sometimes } \operatorname{CaCl} X)$$

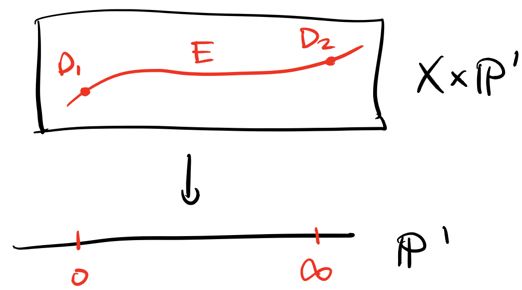
Rmk: $\underset{\{\mathcal{O}(D)\}}{\overset{\shortparallel}{\operatorname{Cl} X}} \subset \underset{\{L \text{ l.b.}\}}{\overset{\shortparallel}{\operatorname{Pic} X}}$ $\qquad$ ($\operatorname{Cl} X = A^1(X)$ Chow group of codim-1 cycles)

Def: $D, D' \in \operatorname{Div} X$ are linearly equivalent (or rat'l equiv) if $\mathcal{O}(D) \cong \mathcal{O}(D')$. Also written $D \sim D'$, $D \sim_{lin} D'$

Explicitly this means: $D' = D + \operatorname{div}(f) \quad f \in K(X)^\times$.

Ex: $X = \mathbb{P}^n$, $\operatorname{Pic} X = \mathbb{Z} = \{\mathcal{O}(d) : d \in \mathbb{Z}\}$
f, g hom pols of degs d & e, then $\operatorname{div} f \sim \operatorname{div} g$
$\iff d = e$

Fact: If $D_1 \sim D_2$ on X, then $\exists\, E \in \mathrm{Div}(X \times \mathbb{P}^1)$ s.th. $E|_{X \times \{0\}} = D_1$, $E|_{X \times \{\infty\}} = D_2$



Ex: $X = \mathbb{P}^1$, Divisors on X with $\mathcal{O}(D) = \mathcal{O}(1)$

$\Leftrightarrow$ deg 1 hom pol $\Leftrightarrow$ points on $\mathbb{P}^1$

$$D = \sum a_i [P_i], \quad \mathcal{O}(D) = \mathcal{O}(\underbrace{\textstyle\sum a_i}_{\text{deg} = \#\text{pts w/ mult}})$$

Néron-Severi group: $NS(X)$

$$\begin{array}{ccccc} H^0(X, K_X^\times/\mathcal{O}_X^\times) & \xrightarrow[\beta]{\mathcal{O}(-)} & H^1(X, \mathcal{O}_X^\times) & \xrightarrow[\gamma]{c_1(-)} & H^2(X, \mathbb{Z}) \\ D & \longmapsto & \mathcal{O}(D) & & \\ & & L & \longmapsto & c_1(L) \end{array}$$

from seq $\mathcal{O}_X^\times \to K_X^\times \to K_X^\times/\mathcal{O}_X^\times$

from exp. seq

$$\mathrm{Div}\, X \xrightarrow{\beta} \mathrm{Pic}\, X \xrightarrow{\gamma} H^2(X, \mathbb{Z})$$

$\mathrm{Im}(\gamma \circ \beta) =: NS\,X$ "divisors up to homological (num.) eq."

$D \sim_{hom} D'$ if their images in $H^2(X, \mathbb{Z})$ coincide.

Sections of line bundles

$$\begin{tikzcd} L \arrow[d, "\pi"'] \\ X \arrow[u, bend left, "s"] \arrow[u, bend right, "0"'] \end{tikzcd} \qquad \begin{aligned} Z(s) = s^{-1}(0) &= \{x \in X : s(x) = 0(x)\} \subset X \\ &= s^{-1}(0(X)) \end{aligned}$$

Prev: $Z(s)$ analytic hypersurface when $s \neq 0$

<u>Now</u>: $Z(s) \in \operatorname{Div} X$ when $s \neq 0$ (if X not conn: $Z(s)$ nowhere dense)

Pick trivialization of L: $X = \cup U_i$, $L|_{U_i} \xrightarrow[\varphi_i]{\cong} U_i \times \mathbb{C} \xrightarrow{pr_2} \mathbb{C}$

$s|_{U_i} \longleftrightarrow f_i : U_i \longrightarrow \mathbb{C}$ holo fcn

$f_i = pr_2 \circ \varphi_i \circ s$

$$\begin{tikzcd} L|_{U_i} \arrow[r, "\varphi_i"] \arrow[dr, "\pi"'] & U_i \times \mathbb{C} \arrow[d, "pr_1"] \\ & U_i \arrow[ul, bend left, "s"] \end{tikzcd}$$

$Z(s) = (U_i, f_i)$

Verify that f_i/f_j nowhere vanishing holo fcn. But f_i/f_j "$=$" $\varphi_i \circ \varphi_j^{-1}$ "$=$" ψ_{ij} the transition fcns of L and they lie in $\Gamma(U_{ij}, \mathcal{O}_X^\times)$.

<u>Rmk</u>: $\mathcal{O}(Z(s)) \cong L$.

$$\begin{tikzcd} \operatorname{Div} X \arrow[r] & \operatorname{Pic} X \arrow[l, bend right, dashed] \end{tikzcd}$$

$$D \longmapsto \mathcal{O}(D)$$

$$Z(s) \longleftarrow\!\shortmid (L, s)$$

"Divisors are line bundles w/ sections"

Not quite true: e.g. on $\mathbb{P}^n$, $\mathcal{O}(-1)$ has no sections.

but $-Z(\text{hom pol of deg } 1) = -[\text{hyperplane}]$

Solution: Meromorphic sections of l.b.

Def: A m.m. section of L is:

- an open cover $X = \bigcup U_i$
- a trivialization of L: $L|_{U_i} \xrightarrow[\varphi_i]{\cong} U_i \times \mathbb{C}$
- meromorphic fcns $f_i \in K(U_i)$

s.th. $f_i = \varphi_{ij} f_j$

$$(U_i, f_i)$$

A meromorphic section $s \neq 0$ of L gives $Z(s) \in \operatorname{Div} X$

Rmk: A m.m. section s of L is a divisor $D \in \operatorname{Div} X$ s.th. $\mathcal{O}(D) \cong L$

So $\operatorname{Div} X = \{(L, s) : L \in \operatorname{Pic} X,\ s \text{ meromorphic section}\}$

Rmk: A m.m. section of $\mathcal{O}$ is just a m.m. function

Rmk: $\Gamma(X, \mathcal{O}_X) \setminus \{0\}$ non-zero holo fcn monoid

$K(X)^\times$ group

$$K(X)^\times \xrightarrow{\text{div}} \operatorname{Div} X \quad \text{group homo}$$

Rmk: $\Gamma(X, L) \setminus \{0\}$ not monoid

$\mathcal{M}(X, L) \setminus \{0\}$ m.m. sections of L not group

$s_1 \in \Gamma(X, L_1)$, $s_2 \in \Gamma(X, L_2)$

$s_1 \otimes s_2 \in \Gamma(X, L_1 \otimes L_2)$ $\qquad Z(s_1 \otimes s_2) = Z(s_1) + Z(s_2)$

Ex: f_1, f_2 hom pols of degrees d_1 & d_2 $\Rightarrow$ $f_1 f_2$ hom pol of deg $d_1 + d_2$