Complex Algebraic Geometry
Lecture #9, Apr 21, 2020
(SF3612)

- Sections of line bundles and divisors (cont)
- Ideal sheaf $\mathcal{O}(-D)$
- Maps to $\mathbb{P}^n$ and sections of line bundles
- Examples

Recall: Divisors

- Weil divisors

$$\mathrm{WDiv}\, X = \{\sum a_i [Y_i] : a_i \in \mathbb{Z},\ Y_i \text{ hypersurface}\}$$

- Cartier divisors

on $U_{ij} = U_i \cap U_j$

$$\mathrm{CDiv}\, X = \{(U_i, f_i) : X = \cup U_i,\ f_i \in K(U_i)^\times,\ f_i = f_j \cdot \psi_{ij},\ \psi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^\times)\} / \sim$$
$$= \Gamma(X, K_X^\times / \mathcal{O}_X^\times)$$

$(U_i, f_i) \sim (U_i, hf_i)$, $h \in \Gamma(X, \mathcal{O}_X^\times)$
and refinements: $X = \cup U_i = \cup V_j$

- Principal divisors:

$$\mathrm{PDiv}\, X = \{\mathrm{div}(f) : f \in K(X)^\times\}$$

$\mathrm{div}(f) = Z(f) - P(f)$ w/ multiplicity

- Weil = Cartier (X mfld)

$\mathrm{CDiv}\, X \xrightarrow{\sim} \mathrm{WDiv}\, X$ group isomorphism

Recall: line bundles

- **Line bundles** $= \{ L \xrightarrow{\pi} X, \overbrace{\text{vsp str. on}}^{\text{of dim } 1} L(x) = \pi^{-1}(x) : \ldots \}$

$$\operatorname{Pic} X = H^1(X, \mathcal{O}_X^\times) = \{(U_i, \varphi_{ij}) : X = \cup U_i, \varphi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^\times)\}/\sim$$

- **Line bundles** of divisors

$$\operatorname{CDiv} X \longrightarrow \operatorname{Pic} X$$

$$D = (U_i, f_i) \longmapsto \mathcal{O}(D) = (U_i, \varphi_{ij}) \qquad \varphi_{ij} = f_i / f_j$$

- **An exact sequence**

$$1 \longrightarrow \operatorname{PDiv} X \longrightarrow \operatorname{CDiv} X \xrightarrow{\alpha} \operatorname{Pic} X, \quad \text{ie, } \ker(\alpha) = \operatorname{PDiv} X$$

$$D \longmapsto \mathcal{O}(D)$$

- **Class group** (divisors up to linear eq.) $\qquad f \in K(X)^\times$

$$\operatorname{Cl} X = \operatorname{CDiv} X / \underbrace{\operatorname{PDiv} X}_{= \operatorname{im} \alpha} \qquad D \sim E \iff D = E + \operatorname{div}(f)$$

$$\iff \mathcal{O}(D) \cong \mathcal{O}(E)$$

$$\operatorname{Cl} X \xrightarrow{\bar{\alpha}} \operatorname{Pic} X$$

$$D \longmapsto \mathcal{O}(D)$$

- **Néron – Severi group** (divisors up to homological eq.)

$$\operatorname{CDiv} X \longrightarrow \underset{H^1(X, \mathcal{O}_X^\times)}{\overset{\shortparallel}{\operatorname{Pic} X}} \xrightarrow{c_1} \!\!\!\!\!\rightarrow NS(X) \subset H^2(X, \mathbb{Z})$$

$$D \longmapsto \mathcal{O}(D)$$

$$L \longmapsto c_1(L)$$

Sections of line bundles

Holomorphic sections: $\begin{tikzcd} L \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \arrow[u, bend right, "0"'] \end{tikzcd}$ $\rightsquigarrow$ divisor $Z(s)$

As a set:

$$Z(s) = s^{-1}(0(X)) = \{x \in X : \underset{\substack{\uparrow \\ L(x) \cong \mathbb{C} \\ \text{choice}}}{s(x)} = 0\}$$

As a Cartier divisor: $X = \cup U_i$, $L|_{U_i}$ trivial, $s|_{U_i} = \underset{\text{holo fcn}}{f_i}$

$Z(s) := (U_i, f_i)$ defined up to mult w/ $\Gamma(U_i, \mathcal{O}_X^\times)$

<u>Note</u>: $f_i/f_j = \psi_{ij}$ transition functions of L

Meromorphic sections: $\begin{tikzcd} L \arrow[d] \\ X \arrow[u, dashed, bend left, "s"] \end{tikzcd}$ + more structure

not in Huybrechts

$$\mathcal{M}(X, L) = \{s = (U_i, f_i) : X = \cup U_i,\ f_i \in K(U_i)^\times \text{ s.th } f_i/f_j = \underbrace{\psi_{ij}}_{\text{transition fcns of } L}\}$$

trivialization of L

<u>If $s \neq 0$:</u>
$Z(s) = (U_i, f_i) \in \mathrm{CDiv}\, X$

If s meromorphic section of L, then by definition $\mathcal{O}(Z(s)) = L$ and:

$$\beta^{-1}(L) = \mathcal{M}(X, L) \setminus 0 / \sim$$

$(U_i, f_i) \sim (U_i, g_i)$ if $\exists h \in \Gamma(X, \mathcal{O}_X^\times)$ s.th. $g_i = h f_i$ and refinements

where

$$\mathrm{CDiv}\, X \xrightarrow{\beta} \mathrm{Pic}\, X$$
$$D \longmapsto \mathcal{O}(D)$$

<u>SLOGAN</u>: A divisor <u>is</u> a line bundle w/ a meromorphic section. (up to mult w/ $\mathcal{O}_X^\times$)

Further consequences

- effective divisors $\sum a_i[Y_i]$ $a_i \geq 0$ $\Longleftrightarrow$ (U_i, f_i) $f_i \in \Gamma(U_i, \mathcal{O}_X^\times)$ $\Longleftrightarrow$ line bundle w/ a holo. section.

  holo and not mero.

- β need not be surjective (image is $Cl(X)$)

  image consists of line bundles admitting a non-zero mero. section.

- Fact: β is surjective if X is projective.

If we start with $D \in CDiv X$, $\rightsquigarrow L = \mathcal{O}(D) \in Pic X$

$D \longleftrightarrow (L, s_D)$

+ meromorphic section s_D of L well-defined up to mult w/ $\Gamma(X, \mathcal{O}_X^\times)$

(holo if D effective)

Ideal sheaf of a divisor

Recall: $Y \subset X$ submanifold, $\mathcal{I}_Y \subset \mathcal{O}_X$ sheaf of holo fcns vanishing on Y:

$$\mathcal{I}_Y(U) = \{ f : U \to \mathbb{C} \text{ holo} : f(y) = 0 \ \forall y \in Y \}.$$

Exact sequence:

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0$$

Today: for divisor D want $\mathcal{I}_D \subset \mathcal{O}_X$ remembering multiplicities.

Ex: $D = 2[0]$ on $X = \mathbb{C}$, $D = Z(z^2)$. Want:

$\mathcal{I}_D = (z^2) = \{ f : \mathbb{C} \to \mathbb{C} : f(0) = 0, f'(0) = 0 \}$ = holo fcns vanishing at least twice at 0.

First suppose: $D = [Y]$ Y irreducible hypersurface on X

$\iff$ locally $Y = Z(f)$ f holo fcn

$\iff$ locally $\mathcal{I}_Y = (f)$

$D \iff L = \mathcal{O}(D)$, $s_D \in \Gamma(X, L)$ holo section

$\iff \mathcal{O}_X \xrightarrow{s_D} L$ ($\mathcal{O}_X$-linear map so defined by image of 1)

$f \longmapsto f \cdot s_D$

$\iff L^\vee \xrightarrow{s_D^\vee} \mathcal{O}_X$

$g \longmapsto g \cdot s_D$

$\mathcal{O}(-D) = L^\vee$

if $D = (U_i, f_i)$ $\varphi_{ij} = f_i/f_j$, L

then $-D = (U_i, \frac{1}{f_i})$ $\varphi_{ij} = f_j/f_i$, $L^\vee$

$$\boxed{\mathcal{O}(-D) \xrightarrow{s_D^\vee} \mathcal{O}_X \quad (*)}$$

<u>Lemma 2.3.22</u>: $s_D^\vee$ injective and image is $\mathcal{I}_Y$.

<u>pf</u>: Can verify this locally. Then $Y = Z(f)$, $\mathcal{I}_Y = (f)$

$L \cong \mathcal{O}_X$. $s_D^\vee : \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is mult w/ f.

$(*)$ becomes $\mathcal{O}_X \xrightarrow{\cdot f} \mathcal{O}_X$. It is injective because f is invertible on dense open $X \setminus Y$. (if $fg = 0 \Rightarrow g|_{X \setminus Y} = 0 \Rightarrow g = 0$) $\square$

For general effective $D = \sum a_i [Y_i]$, $a_i \geq 0$, we have

$$\mathcal{O}(-D) \xrightarrow{s_D^\vee} \mathcal{O}_X.$$

Same proof shows $s_D^\vee$ is injective and we <u>define</u> $\mathcal{I}_D := \operatorname{im}(s_D^\vee)$.

<u>Define</u>: $\mathcal{O}_D := \mathcal{O}_X / \mathcal{I}_D$ "holo fcns on D" (D not nec a manifold)

Maps to $\mathbb{P}^N$ and linear systems

<u>Motivation</u>: $s \in \Gamma(X, \mathcal{O}_X) \iff f: X \to \mathbb{C}$ holo fcn

$s_1, \dots, s_N \in \Gamma(X, \mathcal{O}_X) \iff f_1, \dots, f_N : X \to \mathbb{C}$ holo fcns

$\iff f : X \to \mathbb{C}^N$

$x \mapsto (f_1(x), \dots, f_N(x))$

Given $s_0, s_1, \dots, s_N \in \Gamma(X, L)$, define map:

$$\begin{tikzcd} L \arrow[d, shift left] \\ X \arrow[u, shift left, "s_i"] \end{tikzcd}$$

$$X \setminus Bs \longrightarrow \mathbb{P}^N = (\mathbb{C}^{N+1} \setminus \{0\}) / \mathbb{C}^\times$$

$$x \longmapsto (s_0(x) : s_1(x) : \dots : s_N(x))$$

$$= (\lambda s_0(x) : \lambda s_1(x) : \dots : \lambda s_N(x))$$

$s_i(x) \in \underbrace{L(x)}_{\text{1-dim v.sp.}} \cong \mathbb{C}$

defined up to mult. with $\lambda \in \mathbb{C}^\times$

Map only defined where at least one $s_i(x) \neq 0$.

<u>Def</u>: $x \in X$ is a **base point** of $\{s_0, \dots, s_N\}$ if $s_0(x) = \dots = s_N(x) = 0$.

The **base locus** $Bs(s_0, \dots, s_N)$ is the set of base points.

<u>Def</u>: A **linear system** is a finite dim'l linear subspace $V \subset \Gamma(X, L)$.

$Bs(V) = \{ x \in X : s(x) = 0 \ \ \forall s \in V \}$.

A **complete linear system** is $V = \Gamma(X, L) =: |L|$

If D is a divisor, then $|D| := |\mathcal{O}(D)|$.

<u>Fact</u>: X projective $\Rightarrow \Gamma(X, L)$ finite-dimensional v. sp.

Def: Let V be a linear system, with basis $s_0, \dots, s_N$.

$$\varphi_V : X \setminus \mathrm{Bs}\, V \longrightarrow \mathbb{P}^N$$
$$x \longmapsto (s_0(x) : \cdots : s_N(x))$$

Basis-free version:

$$X \setminus \mathrm{Bs}\, V \longrightarrow \mathbb{P}(V^\vee) = (V^\vee \setminus \{0\}) / \mathbb{C}^\times$$
$$x \longmapsto (s \longmapsto s(x))$$

Prop 2.3.26: (i) φ_V holomorphic map and

(ii) $\varphi_V^* \mathcal{O}_{\mathbb{P}^N}(1) \cong L|_{X \setminus \mathrm{Bs}V}$

pf: (i) Seen φ_V map. Holom. can be checked locally.
Then s_i becomes holo. functions (up to a choice).

$$\varphi_V : X \setminus \mathrm{Bs}V \xrightarrow{\text{holo}} \mathbb{C}^{N+1} \setminus \{0\} \xrightarrow{\text{holo}} \mathbb{P}^N.$$

(ii) WLOG X connected. Can assume (reorder) s_0 is not identically 0.
$\mathcal{O}_{\mathbb{P}^N}(1)$: sections are linear polynomials, corr div. are hyperplanes

$H := Z(z_0)$, $\quad \mathcal{O}(H) = \mathcal{O}(1)$

$f^* H = Z(s_0)$ divisor with $\underset{\substack{\| \\ L}}{\mathcal{O}(f^* H)} = \underset{\substack{\| \\ f^*\mathcal{O}(1)}}{f^* \mathcal{O}(H)}$ $\square$

Pull-back of line bundles & divisors along $f : X \to Y$

$$\underset{\substack{\in \\ \mathrm{Pic}\, Y}}{(U_i, \varphi_{ij})} \longmapsto \underset{\substack{\in \\ \mathrm{Pic}\, X}}{(f^{-1}(U_i), \underset{\substack{\| \\ \varphi_{ij} \circ f}}{f^* \varphi_{ij}})}$$

and sim. for divisors

Examples

Ex (elliptic curve) $\underset{a\mathbb{Z}+b\mathbb{Z}}{\overset{}{\Lambda}} \subset \mathbb{C} \quad (\mathbb{Z}+\tau\mathbb{Z})$ $\qquad X = \mathbb{C}/\Lambda$

HW #2 game:

$$X \longrightarrow \mathbb{P}^2$$
$$x \longmapsto \begin{cases} (\wp(x) : \wp'(x) : 1) & \text{if } x \neq 0+\Lambda \\ (0 : 1 : 0) & \text{if } x = 0+\Lambda \end{cases}$$

Weierstrass function:

$$\wp(x) = \frac{1}{x^2} + \sum_{\omega \in \Lambda \setminus 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \quad \text{poles of ord 2 at } \Lambda.$$

$$\wp'(x) = \frac{-2}{x^3} + \sum_{\omega \in \Lambda \setminus 0} \frac{-2}{(z-\omega)^3} \quad \text{poles of ord 3 at } \Lambda$$

Let $\underset{\uparrow \text{ oh}}{O} = \underset{\uparrow \text{ zero}}{0} + \Lambda$ point of X, "origin" divisor $D = 3[O]$

line bundle $L = \mathcal{O}(3[O])$

$s_D \in \Gamma(X, L)$

Multiplying w/ s_D we get holo sections:

$$\wp(x)s_D,\ \wp'(x)s_D \in \Gamma(X, L)$$

and linear system:

$$V = \langle \wp(x)s_D, \wp'(x)s_D, s_D \rangle \subseteq \Gamma(X, L)$$

with base locus:

$$\mathrm{Bs}\, V = \underbrace{Z(s_D)}_{\text{only vanish at } O} \cap \underbrace{Z(\wp'(x)s_D)}_{\text{doesn't vanish at } O} = \emptyset$$

The linear system gives a holo map:

$$\varphi_V : X \longrightarrow \mathbb{P}^2$$
$$x \longmapsto (\wp(x)s_D : \wp'(x)s_D : s_D)$$

when $x \neq 0$, then $s_D(x) \neq 0$ and

$$\varphi_V(x) = (\wp(x) : \wp'(x) : 1)$$

when $x = 0$, then $\wp(x)s_D(x) = 0$ and $\wp'(x)s_D(x) \neq 0$ and

$$\varphi_V(x) = (0 : \wp'(x)s_D(x) : 0) = (0:1:0)$$

Ex (Veronese) $X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(d)$

$V = |L|$ = hom. pols of deg d

basis $z_0^d, z_0^{d-1}z_1, \ldots, z_n^d$ $\qquad N = \dim V - 1 = \binom{n+d}{n} - 1$

$$\mathbb{P}^n \xrightarrow{\varphi_V} \mathbb{P}^N \qquad \underline{\text{Veronese map}}$$
$$z \longmapsto (z_0^d : z_0^{d-1}z_1 : \cdots : z_n^d)$$

Exc: φ_V embedding onto a closed submanifold.

Ex: $n=1$, $d=2$

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2$$
$$(z_0 : z_1) \longmapsto (z_0^2 : z_0 z_1 : z_1^2) = (w_0 : w_1 : w_2)$$

We note that $w_0 w_2 = w_1^2$ and in fact, can show that:

$$\mathbb{P}^1 \cong Z(w_0 w_2 - w_1^2) \subset \mathbb{P}^2$$