Intro (David Rydh):

I Derived categories

II Derived algebraic geometry

III ∞-categories

I Derived categories:

X/k scheme, $\mathcal{F} \in QCOH(X)$

$\bigoplus_k H^k(X, \mathcal{F})$ gr. k-vector space

$$f : X \longrightarrow Y$$

$R^i f_* (\mathcal{F}) \in QCOH(Y)$

Calculate: $\mathcal{F} \longrightarrow I^\bullet$ inj. resolution

$f_* I^n$ cochain cmplx

$$R^i f_* \mathcal{F} = H^i(f_* I^\bullet)$$

Derived push-forward: $Rf_* \mathcal{F} := f_* I^\bullet$ well-def. up to quasi-iso. More structure!

$$\begin{tikzcd} \mathbb{R}f_* : D(\mathrm{QCoh}\, X) \arrow[r] & D(\mathrm{QCoh}\, Y) \arrow[d, "\mathcal{H}^k"] \\ \mathrm{QCoh}\, X \arrow[u] \arrow[r, "R^k f_*"'] & \mathrm{QCoh}\, Y \end{tikzcd}$$

$\mathcal{F}[0] \leftarrow\!\shortmid \mathcal{F}$

<u>Construction</u>: $\overset{\text{ab. cat.}}{\mathcal{A}} \rightsquigarrow Ch(\mathcal{A}) \xrightarrow[\text{q-iso.}]{\text{invert}} D(\mathcal{A})$

$(\mathrm{Mod}\, R,\ \mathrm{QCoh}\, X, \ldots)$

$$\left[\begin{array}{c} X \xrightarrow{f} Y \xrightarrow{g} Z \\ \\ \mathbb{R}g_* \mathbb{R}f_* \simeq \mathbb{R}(gf)_* \end{array}\right]$$

<u>Natural objects in $D(\mathrm{QCoh}\, X)$</u>:

(i) <u>Cotangent complex</u>

$X \longrightarrow Y \rightsquigarrow L^{\bullet}_{X/Y} \in D^{\leq 0}(\mathrm{QCoh}\, X)$

<u>Ex</u>: $X \xrightarrow{\text{smooth}} \mathrm{Spec}\, k$, $\quad L^{\bullet}_{X/k} = \Omega_{X/k}[0]$ v.b.

X l.c.i. $\quad L^{\bullet}_{X/k} \in D^{[-1,0]}(X)$

$X \xhookrightarrow{I} Y$ smooth $\qquad L^\bullet_{X/Y} = [I/I^2 \to \Omega_Y|_X]$

$I = (f_1, \dots, f_n)$ reg. seq. $\qquad \mathcal{H}^0(L^\bullet_{X/Y}) = \Omega_{X/Y}$

$$\boxed{\underline{\text{In general}} \quad \mathcal{H}^0(L^\bullet_{X/Y}) = \Omega_{X/Y}}$$

the cotangent complex controls deformations and obstructions.

Ex:

$$\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, "\text{flat}"'] & X \arrow[d, dashed] \\ Y_0 \arrow[r, hook, "I"] & Y \end{tikzcd}$$

$o(X_0/Y_0, Y_0 \hookrightarrow Y) \in \mathrm{Ext}^2(L^\bullet_{X_0/Y_0}, f^*I)$

$I^2 = 0$

If X_0/Y_0 smooth

$$\text{obs} = H^2(\Omega^\vee_{X_0/Y_0} \otimes f^*I)$$

<u>(ii) Dualizing complexes / Grothendieck duality:</u>

<u>Serre duality:</u> X/k smooth, dim d, perfect pairing

$$H^d(X, \mathcal{F}) \otimes H^{n-d}(\mathcal{F}, \omega_X) \longrightarrow H^n(X, \omega_X)$$

$$\omega_X = \Lambda^d \Omega_X.$$

X not smooth: dualizing complex

$$\omega_X^\bullet \in D^{[-d,0]}(\mathrm{QCoh}\, X)$$

$$\omega_X^\bullet \cong (\Lambda^d \Omega_X)[d]$$

$\omega_X^\bullet \cong \omega_X[d]$ if X Cohen-Macaulay.

Only exists in derived category.

$f : X \longrightarrow Y$, $f^! : D(Y) \longrightarrow D(X)$

f proper: Rf_*, $f^!$ adjoint

(Rf^*, Rf_* — " —)

$D(X)$ interesting "invariant" of X

$D(X) \cong D(Y)$?

Fourier-Mukai ——

Semi-orth. decomp.

Ex: $D(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$

$D(\mathrm{Bl}_Z X)$ build up from $D(X)$ and $D(Z)$.

$\mathrm{QCoh}\, X \subset D(\mathrm{QCoh}\, X)$

Heart of a t-structure

Vary t-structure $\mathcal{A} \subset D(\mathrm{QCoh}\, X)$ other interesting abelian category. (Bridgeland ...)

Ex:

DT-theory: moduli of quot. sheaves

$$\mathcal{I} \subset \mathcal{O}_X \twoheadrightarrow \mathcal{F}$$

PT-theory: moduli of stable pairs

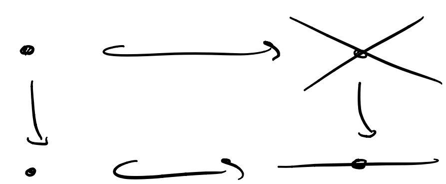
$$\mathcal{O}_X \xrightarrow{f} \mathcal{F}$$

coker f finite.

II DAG:

$$\begin{tikzcd}
\begin{pmatrix}\text{reduced}\\ \text{schemes}\end{pmatrix} \arrow[r, hook, "\text{f.f.}"'] & (\text{schemes}) \arrow[l, bend right, "\text{red.}"'] \arrow[r, hook, "\text{f.f.}"'] & \begin{pmatrix}\text{Derived}\\ \text{schemes}\end{pmatrix} \arrow[l, bend right, "\heartsuit"']
\end{tikzcd}$$

$X \times_Z Y$ may be non-reduced even if X, Y, Z reduced.



Claim: fiber prod. in (Sch) is bad unless tor-indep.
e.g. if f flat.

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

Ex:
- codim. may decrease
- $Lg^* Rf_* \longrightarrow Rf'_* Lg'^*$ not nec. iso.
-
- f reg. emb $\not\Rightarrow$ f' reg. emb.

Ex:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "{(x^2, xy, y^2)}"', "\text{not reg. emb. codim 2.}"] & 0 \arrow[d, "\text{emb. codim. 3}"] \\
\mathbb{A}^2 \arrow[r, "{x^2, xy, y^2}"'] & \mathbb{A}^3
\end{tikzcd}$$

Ex:

$$\begin{tikzcd}
Z_2 \cap Z_1 \arrow[r] \arrow[d] & Z_1 \arrow[d, "\text{smooth}"] \\
Z_2 \arrow[r, "\text{smooth}"'] & X
\end{tikzcd}$$

could be very bad

Ex:

$$\begin{tikzcd}
Z \arrow[r, "="] \arrow[d, "="'] & Z \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

Derived fiber product better:
- pres. reg. emb.
- pres. codim.
- pres. cohomg. complex.

Ex:

$$\begin{tikzcd} & 0 \arrow[d, hook] \\ 0 \arrow[r, hook] & \mathbb{A}^2 \end{tikzcd}$$

$$A/(x,y) = k \twoheadleftarrow k[x,y] = A$$

<u>Koszul resolution</u> $B^\bullet := [\underset{-1}{A} \xrightarrow{\cdot x} \underset{0}{A}] \otimes [A \xrightarrow{\cdot y} A]$

$$= [\underset{-2}{A} \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} \underset{-1}{A^2} \xrightarrow{[x \ y]} \underset{0}{A}]$$ ~~dga!~~

$$\mathcal{H}^k(B^\bullet) = \begin{cases} A/(x,y) & k=0 \\ 0 & o/w \end{cases}$$

Proj. resolution of $A/(x,y)$.

$$A/(x,y) \overset{\mathbb{L}}{\otimes} A/(x,y) = B^\bullet \otimes A/(x,y) = [k \xrightarrow{0} k^2 \xrightarrow{0} k]$$

$$0 \underset{\mathbb{A}^2}{\overset{der}{\times}} 0 = \mathrm{RSpec}([k \xrightarrow{0} k^2 \xrightarrow{0} k]) \quad \left\{ \begin{matrix} e_1^2 = e_2^2 = 0 \\ e_1 e_2 = 1 \end{matrix} \right\}$$

$$L_{w/.} = k^2[-1] \qquad \text{"codim. 2 reg. emb."}$$

<u>Serre's Tor-formula:</u> X smooth of dim d, Z_1, Z_2 of dim d_1, d_2 s.t. $d_1 + d_2 = d$. Suppose $Z_1 \cap Z_2$ has dim 0.

<u>Ex:</u> $C_1, C_2 \subset \mathbb{P}^2$ of deg. n_1, n_2

$C_1 \cap C_2$ $n_1 n_2$ points counted w. mult.

$$\mathrm{mult}_p(C_1 \cap C_2) = \dim_k(\mathcal{O}_{C_1 \cap C_2, p})$$

<u>Beszout's Thm:</u>

$$\sum_{p \in C_1 \cap C_2} \mathrm{mult}_p(C_1 \cap C_2) = n_1 n_2$$

<u>Serre's correction:</u>

$$\mathrm{mult}_p(Z_1, Z_2) = \sum_{n \geq 0} (-1)^n \dim_k(\mathrm{Tor}_k^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})) = \chi(\mathcal{O}_{Z_1 \cap^{\mathrm{der}} Z_2})$$

$$\mathrm{Tor}_0^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = \mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_2} = \mathcal{O}_{Z_1 \cap Z_2}$$

Derived: $$\mathcal{H}^k(\mathcal{O}_{Z_1 \times_X^{\mathrm{der}} Z_2}) = \mathcal{H}^k(\mathcal{O}_{Z_1} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{O}_{Z_2})$$

$$= \mathrm{Tor}_k^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})$$

$$\begin{tikzcd} & Z \arrow[d, hook] \\ D \arrow[r, hook] & X \end{tikzcd}$$ Cartier divisor

$[D] \cap [Z] = [D \cap Z]$ if $D \cap Z \hookrightarrow D$ Cartier divisor
$ \rotatebox{90}{$\in$} \, A^2(X)$

$c_1(\mathcal{O}(D)) \cap [Z] = c_1(\mathcal{O}(D)|_Z) \in A^1(Z)$

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & X \arrow[d, "s"] \\ X \arrow[r, "0"] & \mathcal{O}(D) \end{tikzcd}$$

$D \rightsquigarrow s \in \Gamma(X, \mathcal{O}(D))$
$\mathcal{O}(D) \rightsquigarrow$ forget s

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "0"] \\ X \arrow[r, "0"] & \mathcal{O}(D) \end{tikzcd}$$

W remembers $\mathcal{O}(D)$.

Often: Singular moduli space $\mathcal{M}$ has nice derived scheme $\mathcal{M}^{der}$

$\mathbb{L}_{\mathcal{M}^{der}} \longrightarrow \mathbb{L}_{\mathcal{M}}$ induces perfect obstruction theory.

III ∞-categories:

Derived / triangulated categories are bad.

Can glue in QCoh X : $X = \bigcup_i U_i$
Can't ——— " ——— $D(X)$.

QCoh(der. sch) are like D(sch).

Solution: Can make $D(A)$ into a stable ∞-category.
Have nice properties like gluing.

Models:

- Quasi-categories
- dg-categories
- simplicially enriched categories
- top. enriched categories.

Hom(A,B) not just a set.
$\pi_0(\mathrm{Hom}(A,B))$ usual Hom's.

$\mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$

$\mathrm{Hom}^k(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \left\{ \begin{array}{l} \text{maps of deg. } k \\ \text{b/w gr. algebras} \end{array} \right\}$

cochain cmplx

$\mathcal{H}^k(\mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = \left\{ \begin{array}{l} \text{maps b/w cmplx} \\ \text{of deg } k \end{array} \right\}$