(Tilman)

Cotangent complex

(1) Kähler differentials
(2) Cotangent complex
(3) Properties
(4) Vistar

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(1) Let k be a ring, A a k-algebra, M an A-module.

Def: A (k-lin.) derivation from A into M is a k-linear map

$$A \longrightarrow M$$

s.t. $$d(ab) = a\,d(b) + d(a)\,b$$

$$(\Rightarrow d|_k = 0)$$

and the set of such is denoted by $\mathrm{Der}_k(A,M)$.

Fact: The functor $M \longmapsto \mathrm{Der}_k(A,-)$ is represented by an A-mod. $\Omega_{A/k}$ and there is a universal derivation

$$d : A \longrightarrow \Omega_{A/k}$$

adjoint to $\mathrm{id}_{\Omega_{A/k}}$.

$$\mathrm{Der}_k(A, M) = \mathrm{Alg}_{k/A}(A, A \oplus M)$$

Ex: $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m) \longrightarrow$

$$\Omega_{A/k} = \langle dx_1, \dots, dx_n \rangle / \left\langle \frac{\partial f_i}{\partial x_j} dx_j \right\rangle_{i,j}.$$

Given $f : A \longrightarrow B \rightsquigarrow$

$$\mathrm{Der}_k(A, M) \xrightarrow{\circ f} \mathrm{Der}_k(B, M)$$

$$\rightsquigarrow$$

$$\begin{tikzcd} \Omega_{A/k} \arrow[r] & \Omega_{B/k} \\ A \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & B \arrow[u] \end{tikzcd} \rightsquigarrow \quad \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k}$$

Given $k \to k' \rightsquigarrow \Omega_{A/k} \to \Omega_{A/k'}$

Prop: Given ring maps $k \to A \to B$ the sequence

$$(0 \longrightarrow)\ \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow \Omega_{B/A} \longrightarrow 0$$

is exact (if B/A $\overset{\text{diff.}}{\text{smooth}}$).

(2) The cotangent complex

$$\begin{tikzcd} \mathrm{sAlg}_k \arrow[d] \\ \mathrm{sMod}_k \arrow[r, "N"', "\text{(Dold-Kan)}\ \simeq"] & \mathrm{Ch}_k^{\geq 0} \end{tikzcd}$$

sAlg_k — simpl. k-algebras = simpl. obj. in Alg_k

$$N(M_\bullet)_n = M_n \big/ (s_i(M_{n-1}))_i$$

$$d : N(M_\bullet)_n \longrightarrow N(M_\bullet)_{n-1}$$

$$d = \sum_{i=0}^{n} (-1)^i d_i$$

We have a model structure on $\mathrm{Ch}_k^{\geq 0}$ where

W = H_*-iso.
F = surjections in pos. deg.
C = (induced from $\circlearrowright$)

Hence we also get a model str. on sMod_k.

W = π_*-equiv.
F = "componentwise" surjections
C = (induced)

We obtain an induced model structure on $sAlg_k$ sharing W and F with $sMod_k$.

Def: A <u>free extension</u> of simpl. algebras is a map

$$A \longrightarrow B \quad \text{s.t.}$$

(1) $B_n = A_n[S_n]$ for a set S_n

(2) $s_j(S_n) \subseteq S_{n+1}$ "degeneracy-free"

Prop: Cofibrations in sAlg are exactly the retracts of free extension.

Ex: If $S_\bullet$ a simpl. set then $k[S_\bullet]$ is a cofibrant simpl. k-alg.

Def: The <u>cotangent complex</u> of A/k is the simpl. A-mod

$$\mathcal{L}_{B/A} = \Omega_{P_\bullet/k} \underset{P_\bullet}{\overset{\text{levelwise}}{\otimes}} A$$

where $k \xhookrightarrow{\text{cofib.}} P_\bullet \xrightarrow[\sim]{\text{triv. fibr.}} A$.

well-def. in the homotopy category.

Constructing resolutions

- The bar construction:

$$\begin{array}{c} \vdots \\ \downarrow\downarrow\downarrow\uparrow\uparrow \\ k[k[A]] \\ \downarrow\ \downarrow\ \uparrow\ \overset{[[a]]}{\underset{[a]}{\uparrow}} \\ k \longrightarrow k[A] \longrightarrow A \end{array}$$

remove brackets at level n

 - free
 - $P_\bullet \longrightarrow A$ is a w.eq

Sample computation:

$L_{A/\mathbb{Z}}$ where $A = \mathbb{Z}[x]/(x^n)$

simpl. resolution

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]/(x^n)$$

$$\tilde{P}_\bullet = \left\{ \begin{array}{c} \mathbb{Z}[y] \\ y \mapsto x^n \ \downarrow\downarrow (d_0\ d_1) \ y \mapsto 0 \\ \mathbb{Z}[x] \end{array} \right.$$

This is semi-simpl. and we make it simpl.

by $P_\bullet = \underset{\Delta^{op}_{inj} \to \Delta^{op}}{\mathrm{LKan}}\, \tilde{P}_\bullet$

Explicitly, $P_\bullet \longrightarrow \mathbb{Z}[x]/(x^n)$

$$P_n = \mathbb{Z}[x, y_\sigma : \sigma : [n] \longrightarrow [1]]$$

$$y = y_{id}$$

$$(\Omega_{P_\bullet/\mathbb{Z}})_n = \mathbb{Z}[x, y_\sigma]\langle dx, dy_\sigma \rangle$$

$$L_{A/\mathbb{Z}} = \left(\Omega_{P_\bullet/\mathbb{Z}} \otimes_{P_\bullet} A\right) \cong \mathbb{Z}[x]/(x^n)\langle dx, dy_\sigma \rangle$$

$$NL_{A/\mathbb{Z}} = \begin{array}{ccc} A\langle dy \rangle & \longrightarrow & A\langle dx \rangle \\ dy & \longmapsto & n x^{n-1} dx \end{array}$$

$$\pi_0 L_{A/\mathbb{Z}} \cong \Omega_{A/\mathbb{Z}} = \left(\mathbb{Z}[x]/(x^n)\right)\langle dx \rangle / (n x^{n-1} dx)$$

$$\pi_1 L_{A/\mathbb{Z}} = H_1 N L_{A/\mathbb{Z}} = xA\langle dy \rangle / (x^n\, dy) \cong A/(x^{n-1})$$

$$\begin{tikzcd}
\vdots & \\
x^2dy \bullet & \\
xdy \bullet & \\
dy \bullet \arrow[r, "n"] & \bullet\, x^{n-1}dx \\
& \vdots \\
& \bullet\, x^2dx \\
& \bullet\, xdx \\
& \bullet\, dx & A\langle dx \rangle
\end{tikzcd}$$