LOCAL STRUCTURE OF ALGEBRAIC STACKS: EXERCISES
LECTURE 2: THE LOCAL STRUCTURE OF ALGEBRAIC STACKS

Exercise 2.1 (Local structure of DM-stacks)**.** Let k be an algebraically closed field and $\mathscr{X}$ a Deligne–Mumford stack of finite type over k with coarse space $\mathscr{X} \to X$. Let $x \in \mathscr{X}(k)$. Prove the local quotient structure: "there exists an étale stabilizer-preserving morphism $([W/G_x], u) \to (\mathscr{X}, x)$" as follows:

(a) Reduce to X henselian local.
(b) Prove that there is a *finite* étale presentation $p\colon U \to \mathscr{X}$ with U affine.
(c) If p has degree d, show that $\mathscr{X} = [V/\Sigma_d]$ for suitable affine V.
(d) Pick a point $v \in V$ above $x \in |\mathscr{X}|$. Show that G_x acts on a clopen subscheme $W \subseteq V$ such that $\mathscr{X} = [W/G_x]$.
(e) When k is not algebraically closed, show that steps (a)–(c) make sense and gives $([V/G], v) \to (\mathscr{X}, x)$ étale, stabilizer-preserving and such that $\kappa(v) = \kappa(x)$.

Definition 2.2. Let $\mathscr{X}$ be a (qcqs) stack and $\mathscr{X}_0$ a closed substack. We say that

- $(\mathscr{X}, \mathscr{X}_0)$ is *local* if every closed non-empty subset of $|\mathscr{X}|$ meets $|\mathscr{X}_0|$.
- $(\mathscr{X}, \mathscr{X}_0)$ is *henselian* if for every finite morphism $\mathscr{X}' \to \mathscr{X}$
$$\mathrm{ClOpen}(\mathscr{X}') \to \mathrm{ClOpen}(\mathscr{X}' \times_{\mathscr{X}} \mathscr{X}_0)$$
is bijective.
- $(\mathscr{X}, \mathscr{X}_0)$ is (coherently) *complete* if $\mathscr{X}$ is noetherian and
$$\mathrm{Coh}(\mathscr{X}) \to \varprojlim_n \mathrm{Coh}(\mathscr{X}_n)$$
is an equivalence of categories ($\mathscr{X}_n$ is the nth infinitesimal neighborhood of $\mathscr{X}_0$).

Exercise 2.3 (Nakayama's lemma for stacks)**.** Suppose that $(\mathscr{X}, \mathscr{X}_0)$ is a local stack.

(a) Show that if $\mathcal{F} \in \mathrm{QCoh}(\mathscr{X})$ is of finite type and $\mathcal{F}|_{\mathscr{X}_0} = 0$, then $\mathcal{F} = 0$.
(b) Show that if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a homomorphism of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{G}$ is of finite type and $\varphi|_{\mathscr{X}_0}$ is surjective, then φ is surjective.

Exercise 2.4 (Complete and henselian pairs I)**.** Let $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ be a closed immersion.

(a) Show that $(\mathscr{X}, \mathscr{X}_0)$ coherently complete $\implies$ $(\mathscr{X}, \mathscr{X}_0)$ henselian $\implies$ $(\mathscr{X}, \mathscr{X}_0)$ local.

Let $p\colon \mathscr{X} \to X$ be a universally closed surjective morphism (e.g., a good moduli space) and let $X_0 = p(\mathscr{X}_0)$.

(b) Show that: $(\mathscr{X}, \mathscr{X}_0)$ local $\implies$ (X, X_0) local.
(c) Show that: $(\mathscr{X}, \mathscr{X}_0)$ henselian $\implies$ (X, X_0) henselian, if p has geometrically connected fibers.

Exercise 2.5 (Complete and henselian pairs II)**.** Let $\mathscr{X}_0 \hookrightarrow \mathscr{X}$ be a closed immersion. Let $p\colon \mathscr{X}' \to \mathscr{X}$ be a morphism and $\mathscr{X}'_0 = p^{-1}(\mathscr{X}_0)$. Give (elementary) proofs of the following facts:

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(1) If $(\mathscr{X}, \mathscr{X}_0)$ is noetherian and henselian and p is proper and representable, then $(\mathscr{X}', \mathscr{X}'_0)$ is henselian. *Hint: use Stein factorizations (these also exists for proper algebraic spaces). In fact, the result is also true without noetherian assumptions.*
(2) If $(\mathscr{X}, \mathscr{X}_0)$ is complete and p is finite, then $(\mathscr{X}', \mathscr{X}'_0)$ is complete.

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